MAU34401 Homework Problem Sheet 3
Submit via Blackboard before 5pm, 18th November 2020 [1]

Policy

Extensions of up to a week will be granted upon request before the due deadline and for sufficient cause. Please include an estimate of the time taken to complete the set and remember to put your name on the solutions.
Please read the following declaration:

I have read and I understand the plagiarism provisions in the General Regulation s of the University Calendar for the current year, found at http://www.tcd.ie/ca lendar.

I have also completed the Online Tutorial on avoiding plagiarism "Ready Steady Write", located at http://tcd-ie.libguides.com/plagiarism/ready-steady-write.

Then sign this sheet and attach it to your solutions before handing them in.

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Name (print) and Signature Date

[1] Sinéad Ryan, see also http://www.maths.tcd.ie/~ryan/3431.html

Just 2 questions for this homework, but question 1 is quite long ... enjoy!

1. Two concentric spheres have radii a_1 and a_2 with $a_2 > a_1$ and each is divided into two hemispheres by the same horizontal plane. The potential on the upper hemisphere of the inner sphere is maintained at constant value $+V$ and the lower hemisphere at zero potential. The potential on the upper hemisphere of the outer sphere is at zero potential while the lower hemisphere is at negative constant value $-V$.

   The general expression for the potential between the two spheres, $a_1 \leq r \leq a_2$, in this case is (using the usual spherical coordinates (r, θ, ϕ))

$$\Phi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} [A_\ell r^\ell + B_\ell r^{-\ell-1}] P_\ell(\cos\theta) \ . \tag{1}$$

   (a) What are the boundary conditions on the potential Φ?

   (b) For $\ell = 0, 1, 2$ use the orthogonality of the Legendre polynomials to derive linear equations for A_ℓ and B_ℓ and solve these explicitly to find the first few terms in the expansion of Φ.

   (c) Find the general expression for arbitrary ℓ.

   (d) Consider now the case where the hemispheres are connected but the inner shell is maintained at zero potential and the outer shell is held at some known fixed potential $V(\theta, \phi)$. Find the potential at every point between the shells as a series of spherical harmonics.

   Hint: for the third question you might like to separately consider the case of even and odd ℓ when integrating the Legendre polynomials by using the Rodgrigues formula in these two scenarios.

2. Consider the vector potential

$$\vec{A}(\vec{x}) = \frac{g}{4\pi} \int_{-\infty}^{0} dz' \ \frac{\hat{z} \times (\vec{x} - z'\hat{z})}{|\vec{x} - z'\hat{z}|^3}.$$

   Interesting fact: This is the Dirac expression for the vector potential of a magnetic monopole located at the origin (and its associated Dirac string along the negative z-axis).

   (a) Caculate $\vec{A}$ explicitly and show that in spherical coordinates it has components

$$A_r = 0, \ A_\theta = 0, \ A_\phi = \frac{g(1-\cos\theta)}{4\pi r \sin\theta} = \frac{g}{4\pi} \frac{\tan\theta/2}{r}.$$

   (b) Calculate $\vec{B}$ and show it has the form of a point charge (ignoring the point at $\theta = \pi$).