1. Consider a conducting sphere, radius a, centred at the origin. A positive charge $+q$ is located at $z = -R$ and a charge $-q$ is located at $z = R$.

   (a) Find a Green's function for the problem and use it to determine the potential outside the sphere.

   The potential on the sphere is zero, which can be seen by calculating the work done in moving a charge from infinity to the surface of the sphere along a line in the xy plane.

   Therefore

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')d^3x' \tag{1}$$

   Remember that V is the volume of interest ie outside the sphere. Noting that the charge distributions due to $+q, -q$ respectively can be written as $q\delta(\vec{x}'+R\hat{k})$ and $-q\delta(\vec{x}' - R\hat{k}$ we have

$$\begin{aligned}\Phi(\vec{x}) &= \frac{1}{4\pi\epsilon_0}\int_V \left(q\delta(\vec{x}'+R\hat{k}) - q\delta(\vec{x}'-R\hat{k})\right)G(\vec{x},\vec{x}')d^3x' \\ &= \frac{q}{4\pi\epsilon_0}\left[G(\vec{x}',-R\hat{k}) - G(\vec{x}',R\hat{k})\right] \qquad (2)\end{aligned}$$

   Now recall that the Green's function for a sphere of radius a was determined in lectures so we use that result here (here I am using the result determined via method of images). The (Dirichlet) Green's function is

$$G(\vec{x},\vec{x}') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2r'^2}{a^2} + a^2 - 2rr'\cos\gamma}}$$

   and applying that result to this problem:

$$G(\vec{r},R\hat{k}) = \frac{1}{R\sqrt{1 - \frac{2r}{R}\cos\theta + \frac{r^2}{R^2}}} - \frac{(a/r)}{R\sqrt{1 - \frac{2a^2}{rR}\cos\theta + \frac{a^4}{R^2r^2}}}.$$

   and

$$G(\vec{r},-R\hat{k}) = \frac{1}{R\sqrt{1 + \frac{2r}{R}\cos\theta + \frac{r^2}{R^2}}} - \frac{(a/r)}{R\sqrt{1 + \frac{2a^2}{rR}\cos\theta + \frac{a^4}{R^2r^2}}}.$$

   Putting these expression together in eqn 2 yields the result.

(b) Show that the charge density on the sphere is $\sigma = 3\epsilon_0 E_0 \cos\theta$ where $E_0 = 2q/(4\pi\epsilon_0 R^2)$.

You can simplify the expression above by considering $R \to \infty$ in which case

$$G(\vec{r}, R\hat{k}) = \frac{1}{R}\left(1 + \frac{r}{R}\cos\theta\right) - \frac{a}{rR}\left(1 + \frac{a^2}{rR}\cos\theta\right)$$

and

$$G(\vec{r}, -R\hat{k}) = \frac{1}{R}\left(1 - \frac{r}{R}\cos\theta\right) - \frac{a}{rR}\left(1 - \frac{a^2}{rR}\cos\theta\right)$$

and combining the terms as previously

$$\Phi(\vec{x}) = -\frac{2q}{4\pi\epsilon_0 R^2}\left(r\cos\theta - \frac{a^3\cos\theta}{r^2}\right)$$

and the charge density on the sphere is $\sigma = \left.\frac{\partial\Phi}{\partial r}\right|_{r=a}$ giving

$$\sigma = 3\epsilon_0 E_0 \cos\theta$$

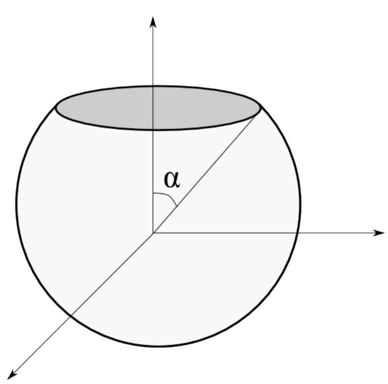
with $E_0 = 2q/4\pi\epsilon_0 R^2$.

2. Consider a spherical shell of radius a with a missing cap at the north pole - defined by the cone with opening angle α and with a uniform charge distribution, σ. The charge distribution in spherical coordinates is

$$\rho(r, \theta, \phi) = \sigma\delta(r - a)\Theta(\cos\alpha - \cos\theta),$$

where $\Theta(x)$ is the Heaviside function [defined by $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$].

Sketch the sphere. Find the potential inside and outside of the spherical surface.



This problem has azimuthal symmetry and so we may use for the potential the general formula

$$\sum_{\ell=0}^{\infty} [A_\ell r^\ell + B_\ell r^{-\ell-1}] P_\ell(\theta) \ . \tag{3}$$

Due to the charge on the sphere we expect the potential to have a jump at $r = a$ so we solve in the two regions $r < a$ and $r > a$ and then match our answers. For the region $r < a$ we demand the solution be well defined at the origin and so $B_\ell = 0$ for all ℓ. In the region $r > a$ we impose that as $r \to \infty$ the potential goes to zero and so $A_\ell = 0$ for all ℓ.

$$\Phi(r,\theta,\phi) = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \ , & r < a \\ \sum_{\ell=0}^{\infty} B_\ell r^{-(\ell+1)} P_\ell(\cos\theta) \ , & r > a \ . \end{cases} \tag{4}$$

We can now demand continuity at the sphere, i.e. $r = a$, such that

$$\sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} B_\ell a^{-(\ell+1)} P_\ell(\cos\theta) \tag{5}$$

and as the P_ℓ are orthogonal we can project onto each coefficient

$$A_\ell = a^{-(2\ell+1)} B_\ell \ . \tag{6}$$

From the Poisson equation

$$\nabla^2 \Phi = -\frac{\sigma}{\epsilon_0} \delta(r-a) \Theta(\cos\alpha - \cos\theta) \tag{7}$$

we can use integration over a region containing the jump

$$\int_{a-\epsilon}^{a+\epsilon} r^2 dr \nabla^2 \Phi = -\int_{a-\epsilon}^{a+\epsilon} r^2 dr \frac{\sigma}{\epsilon_0} \delta(r-a) \Theta(\cos\alpha - \cos\theta) \tag{8}$$

where we have, by integrating the radial derivative term in ∇^2 by parts, and dropping terms which vanish in the limit $\epsilon \to 0$

$$r^2 \frac{\partial \Phi}{\partial r}\bigg|_{a-\epsilon}^{a+\epsilon} = -a^2 \frac{\sigma}{\epsilon_0} \Theta(\cos\alpha - \cos\theta) \ . \tag{9}$$

Hence we find

$$\sum_{\ell=0}^{\infty} B_\ell a^{\ell+2} (2\ell+1) P_\ell(\cos\theta) = -\frac{\sigma}{\epsilon_0} \Theta(\cos\alpha - \cos\theta) \tag{10}$$

and again using the orthogonality of the Legendre functions

$$B_\ell = \frac{a^{\ell+2}\sigma}{2\epsilon_0} \int_{-1}^{\cos\alpha} P_\ell(\cos\theta) d(\cos\theta) \ . \tag{11}$$

To evaluate the integral we can use

$$\frac{dP_{\ell+1}}{dx} - \frac{dP_{\ell-1}}{dx} - (2\ell+1)P_\ell = 0 \ , \quad \ell > 0 \tag{12}$$

which follows from Rodrigues formula and can be found in Jackson Chp 3. Hence we find

$$B_\ell = \frac{a^{\ell+2}\sigma}{2(2\ell+1)\epsilon_0}[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)] \tag{13}$$

where we use $P_\ell(-1) = (-1)^\ell$. This formula is valid for $\ell > 0$ for the case $\ell = 0$ e have $P_0(x) = 1$ so that

$$B_0 = \frac{a^2\sigma}{2\epsilon_0}[\cos\alpha + 1] \ . \tag{14}$$

Hence we find for $r > a$

$$\Phi(r,\theta,\phi) = \frac{a^2\sigma(\cos\alpha+1)}{2r\epsilon_0} + \frac{\sigma}{2\epsilon_0}\sum_{\ell=1}^{\infty}\frac{[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)]}{2\ell+1}\frac{a^{\ell+2}}{r^{\ell+1}}P_\ell(\cos\theta) \ . \tag{15}$$

Let us note that if we let $\alpha \to 0$ so that the shell becomes a sphere the leading term is

$$\Phi(r,\theta,\phi) \to \frac{Q_{\text{tot}}}{4\pi r\epsilon_0} \tag{16}$$

where $Q_{\text{tot}} = 4\pi a^2\sigma$ while the subleading terms vanish which is wh at we would expect. Inside the sphere we have

$$\Phi(r,\theta,\phi) = \frac{a\sigma(\cos\alpha+1)}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0}\sum_{\ell=1}^{\infty}\frac{[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)]}{2\ell+1}\frac{r^\ell}{a^{\ell+1}}P_\ell(\cos\theta) \ . \tag{17}$$

3. The associated Legendre equation (see lecture notes), is a second order differential equation and as such has 2 linearly independent solutions: the *associated Legendre functions of the first and second kind.*

   Denoting these as $P_l^m(x)$ and $\widetilde{P}_l^m(x)$ note that for non-zero m one solution, say $\widetilde{P}_l^m(x)$ diverges as $x \to \pm 1$ and for general m the Rodrigues' formula gives the $P_l^m(x)$ (for $m \geq 0$):

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

   - Comment on the allowed values for m and l so that solutions to the associated Legendre equation remain finite.

- Write down the first few polynomials, specfically $P_0^0(x), P_1^0(x), P_1^1(x), P_2^1(x), P_2^2(x), P_3^1(x), P_3^2(x)$ and use mathematica (or your favourite plotting package) to plot these.

$$\begin{aligned}
P_0^0(x) &= 1 \\
P_1^0(x) &= x \\
P_1^1(x) &= -(1-x^2)^{1/2} \\
P_2^1(x) &= -3x(1-x^2)^{1/2} \\
P_2^2(x) &= 3(1-x^2) \\
P_3^1(x) &= \frac{3}{2}(1-5x^2)^{1/2} \\
P_3^2(x) &= 15x(1-x^2)
\end{aligned}$$

You can see the Legendre polynomials plotted at e.g Wolfram's Mathworld Page

- The Rodriques' formula is also valid for negative m if $|m| \leq l$. Show that $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$.

Consider $(x^2-1)^l = (x+1)^l (x-1)^l$ and then

$$\begin{aligned}
\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l &= \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \left[(x+1)^l (x-1)^l\right] \\
&= \sum_{s=0}^{l+m} \frac{(l+m)!}{(l+m-s)!} \frac{d^{l+m-s}}{dx^{l+m-s}}(x+1)^l \frac{d^s}{dx^s}(x-1)^l
\end{aligned}$$

Now, both s and $l+m-s$ must be $\leq l$, otherwise one of the derivatives is zero: $l+m-s \leq l \Rightarrow s \geq m$. So, for $m \geq 0$ sum runs from $s=m$ to l. For $m<0$ sum runs from $s=0$ to $l+m = l-|m|$. and

$$\frac{d^k (x \pm 1)^l}{dx^k} = \frac{l!}{(l-k)!}(x \pm 1)^{l-k}$$

So,

$$\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l = \sum \frac{(l+m)!}{(l+m-s)!s!} \frac{l!}{(s-m)!}(x+1)^{s-m} \frac{l!}{(l-s)!}(x-1)^{l-s}$$

and so,

$$\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l = (l!)^2 \sum_{s=m}^{l} \frac{(l+m)!(x+1)^{s-m}(x-1)^{l-s}}{(l+m-s)!s!(s-m)!(l-s)!} \quad \text{for} \quad +m$$

$$\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l = (l!)^2 \sum_{s'=0}^{l-m} \frac{(l-m)!(x+1)^{s'+m}(x-1)^{l-s'}}{(l-m-s')!s'!(s'+m)!(l-s')!} \quad \text{for} \quad -m$$

Setting $s = s' + m$

$$\begin{aligned}\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l &= (l!)^2 \sum_{s=m}^{l} \frac{(l-m)!(x+1)^s(x-1)^{l+m-s}}{(l-s)!(s-m)!s!(l+m-s)!} \\ &= \frac{(l-m)!}{(l+m)!}(x^2-1)^m \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l\end{aligned}$$

The last step is to write

$$P_l^m = \frac{-1)^m}{2!l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

and

$$P_l^{-m} = \frac{-1)^{-m}}{2!l!}(1-x^2)^{-m/2}\frac{d^{l-m}}{dx^{l-m}}(x^2-1)^l = \frac{-1)^{-m}}{2!l!}(1-x^2)^{-m/2}\frac{(l-m)!}{(l+m)!}(x^2-1)^m\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$

and the result follows.