MAU34401 Homework Problem Sheet 1 [1]

1. (a) Given two scalar functions Ψ and Φ prove the one-dimensional Green theorem

$$\int_0^1 \left[\Phi\frac{d^2\Psi}{dx^2} - \Psi\frac{d^2\Phi}{dx^2}\right]dx = \left[\Phi\frac{d\Psi}{dx} - \Psi\frac{d\Phi}{dx}\right]\Big|_0^1$$

Consider the first term under the integral on the l.h.s. and use integration by parts

$$\begin{aligned}\int_0^1 \Phi\frac{d^2\Psi}{dx^2}dx &= \int_0^1 \left(\frac{d}{dx}\left[\Phi\frac{d\Psi}{dx}\right] - \frac{d\Phi}{dx}\frac{d\Psi}{dx}\right)dx \\ &= \left[\Phi\frac{d\Psi}{dx}\right]\Big|_0^1 - \int_0^1 \frac{d\Phi}{dx}\frac{d\Psi}{dx}dx\ .\end{aligned}$$

Now repeat this on the second term

$$\int_0^1 \Phi\frac{d^2\Psi}{dx^2}dx = \left[\Phi\frac{d\Psi}{dx} - \Psi\frac{d\Phi}{dx}\right]\Big|_0^1 + \int_0^1 \frac{d^2\Phi}{dx^2}\Psi dx \tag{1}$$

which is the required result.

(b) Now recall the general formula for the potential following from Green's theorem

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{R}d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da' \tag{2}$$

where $R = |\vec{x} - \vec{x}'|$.

Consider the case of a charge free volume, V, enclosed by a sphere of radius R_0 centered on the point $\vec{x}_0$. Specialise the above formula to this case.

Use the divergence theorem to show that

$$\Phi(\vec{x}_0) = \frac{1}{4\pi R_0^2}\int_S \Phi(\vec{x}')da' = \langle\Phi\rangle_S\ . \tag{3}$$

This is the mean value theorem of electrostatics: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

For the case of a charge free volume, V, enclosed by a sphere of radius R_0 centered on the point $\vec{x}_0$

[1] Sinéad Ryan, see also http://www.maths.tcd.ie/~ryan/34401.html

$$\Phi(\vec{x_0}) = \frac{1}{4\pi} \oint_S \left[\frac{1}{R_0} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{R} \right) \Bigg|_{R=R_0} \right] da' \ . \tag{4}$$

where $R = |\vec{x}_0 - \vec{x'}|$ and $R = R_0$ when $\vec{x}'$ lies on the sphere surrounding $\vec{x}_0$ of radius R_0. The normal vector at the surface points out of the volume of interest, in this case in the direction of increasing radius, thus

$$\frac{\partial}{\partial n'} \frac{1}{R} \Bigg|_{R=R_0} = \frac{\partial}{\partial R} \frac{1}{R} \Bigg|_{R=R_0} = -\frac{1}{R_0^2} \ . \tag{5}$$

So we can write

$$\Phi(\vec{x_0}) = -\frac{1}{4\pi R_0^2} \oint_S \Phi(\vec{x}) da' + \frac{1}{4\pi R_0} \oint \vec{\nabla}\Phi \cdot \vec{n} da' \tag{6}$$

Using the divergence theorem one can show that

$$\oint_S \vec{\nabla}\Phi(\vec{x}') \cdot \vec{n} \ da' = -\oint_S \vec{E} \cdot \vec{n} \ da' = \int_V \vec{\nabla} \cdot \vec{E} \ d^3x' = 1/\epsilon_0 \int_V \rho(\vec{x}') \ d^3x' = 0 \tag{7}$$

and hence $\Phi(\vec{x_0}) = \langle \Phi \rangle_S$.

2. Show by direct substitution that

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'.$$

is indeed a solution of the Poission equation $\nabla^2 \Phi = -\rho/\epsilon_0$ as discussed in lectures. You should use spherical coordinates, where the result

$$\nabla^2 \left(\frac{1}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \right),$$

is useful.

Solution appended at the end of this document.

3. (a) Consider the one-dimensional inhomogeneous differential equation

$$\frac{d^2\Psi}{dx^2} + k^2\Psi = -\rho(x)$$

for $k \in \mathbb{R}$ defined in the interval $0 \leq x \leq 1$. Given the Green function, $g(x, x')$, satisfying the equation

$$\frac{d^2 g}{dx^2} + k^2 g = -\delta(x - x')$$

with boundary conditions $g'(0, x') = g'(1, x') = 0$ where $g'(x, x') = \frac{dg(x,x')}{dx}$, show that the general solution is

$$\Psi(x) = \int_0^1 g(x, x')\rho(x')dx' \ ,$$

for homogeneous boundary conditions $\Psi'(0) = \Psi'(1) = 0$. You may use that the Green function is symmetric in its arguments $g(x, x') = g(x', x)$.

We can use the one-dimensional version of Greens theorem given above

$$\int_0^1 \left[\phi\frac{d^2\psi}{dx^2} - \psi\frac{d^2\phi}{dx^2}\right]dx = \left[\phi\frac{d\psi}{dx} - \psi\frac{d\phi}{dx}\right]\Big|_0^1$$

with $\psi(x) = \Psi(x)$ and $\phi(x) = g(x, x')$. The left-hand side is zero using the boundary conditions while using the differential equations satisfied by Ψ and g we have the result

$$\int_0^1 \Big[g(x, x')\rho(x) - \Psi(x)\delta(x - x')\Big]dx$$

as required.

Show that the Green function defined above is given by

$$g(x, x') = \begin{cases} A\cos kx \ , & x < x' \\ B\cos k(1 - x) \ , & x > x' \end{cases}$$

Determine A and B by demanding G be continuous at $x = x'$ and satisfies the *jump condition:* $\lim_{\epsilon\to 0} g'(x' + \epsilon) - g'(x' - \epsilon) = -1$.

Using the general solution for $x \neq x'$

$$g'' = -k^2 g \Rightarrow g = A\cos kx + B\sin kx$$

we have from $g'(x = 0 < x') = 0 \Rightarrow B = 0$ while for $g'(x = 1 > x') = 0 \Rightarrow G'(x > x') = B\cos k(x' - 1)$. Continuity at $x = x'$ implies that

$$A\cos kx' = B\cos k(x' - 1)$$

while the jump condition

$$\lim_{\epsilon \to 0} g'(x' + \epsilon) - g'(x' - \epsilon) = -1$$

gives

$$-kB \sin k(x' - 1) + kA \sin kx' = -1$$

and hence

$$A = -\frac{\cos k(x' - 1)}{k \sin k} \ , \quad B = -\frac{\cos kx'}{k \sin k} \ .$$

(b) Consider the case $k = 0$. Find the coefficients c_n so that

$$g(x; x') = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \sin(n\pi x') \tag{8}$$

is a Green function on the interval $0 \leq x \leq 1$ satisfying Dirichlet boundary conditions at $x' = 0$ and $x' = 1$.

Taking the second derivative of $g(x; x')$ we find that

$$g''(x; x') = -\sum_{n=1}^{\infty} c_n n^2 \pi^2 \sin(n\pi x) \sin(n\pi x') \tag{9}$$

and combining with the sine-series representation of the delta-function

$$\delta(x - x') = 2 \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi x') \tag{10}$$

we find that the defining equation for the Green function is satisfied if $c_n = \frac{2}{\pi^2 n^2}$. As $g(x, x') = 0$ for $x = 0$ and $x = 1$ (or for $x' = 0$ and $x' = 1$) it satisfies Dirichlet boundary conditions.

3431 Solutions : HMW. 1

(2) $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(x')}{|\vec{x}-\vec{x}'|}\, d^3x'$ show $\nabla^2\phi = -\rho/\varepsilon_0$

writing $r = |\vec{x}-\vec{x}'|$

$$\nabla^2\Phi = \frac{1}{4\pi\varepsilon_0}\int_V \nabla^2\left(\frac{1}{r}\right)\rho(x')\, d^3x$$

Note that $\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) = 0$

for $r \neq 0$ and diverges at $r = 0$. ie $\vec{x} = \vec{x}'$

However, we consider the integral of $\nabla^2\left(\frac{1}{r}\right)$ over an arbitrary volume V → and this is finite since,

$$\int_V \nabla^2\left(\frac{1}{r}\right) d^3r = \int_V \nabla\cdot\nabla\left(\frac{1}{r}\right) d^3r = \oint \nabla\left(\frac{1}{r}\right)\cdot\vec{n}\, da$$

$$= -\oint \frac{\vec{r}}{r^3}\cdot\vec{n}\, da$$

$$= -\oint \frac{r\cos\theta\, da}{r^3} = -\oint \frac{\cos\theta\, da}{r^2}$$

and writing da as solid angle

$$= -\oint \frac{1}{r^2}\cdot r^2\, d\Omega = -4\pi$$

so $\nabla^2\left(\frac{1}{r}\right) = 0, \; r \neq 0$

$-\frac{1}{4\pi}\int_V \nabla^2\left(\frac{1}{r}\right) d^3r = 1, \; r = 0$ $\Longrightarrow \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\underline{r}) = -4\pi\delta(\vec{x}-\vec{x}')$

and result follows.