MAU34401 Homework Problem Sheet 1
Submit via Blackboard before 5pm, 16th October, 2020. [1]

1. (a) Given two scalar functions Ψ and Φ prove the one-dimensional Green theorem

$$\int_0^1 \left[\Phi\frac{d^2\Psi}{dx^2} - \Psi\frac{d^2\Phi}{dx^2}\right]dx = \left[\Phi\frac{d\Psi}{dx} - \Psi\frac{d\Phi}{dx}\right]\Bigg|_0^1$$

(b) Now recall the general formula for the potential following from Green's theorem

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{R}d^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]da' \qquad (1)$$

where $R = |\vec{x} - \vec{x}'|$.
Consider the case of a charge free volume, V, enclosed by a sphere of radius R_0 centered on the point $\vec{x}_0$. Specialise the above formula to this case.
Use the divergence theorem to show that

$$\Phi(\vec{x}_0) = \frac{1}{4\pi R_0^2}\int_S \Phi(\vec{x}')da' = \langle\Phi\rangle_S\ . \qquad (2)$$

This is the mean value theorem of electrostatics: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

2. Show by direct substitution that

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}d^3x'.$$

is indeed a solution of the Poission equation $\nabla^2\Phi = -\rho/\epsilon_0$ as discussed in lectures. You should use spherical coordinates, where the result

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right),$$

is useful.

[1] Sinéad Ryan, see http://www.maths.tcd.ie/~ryan/34401.html

3. (a) Consider the one-dimensional inhomogeneous differential equation

$$\frac{d^2\Psi}{dx^2} + k^2\Psi = -\rho(x)$$

for $k \in \mathbb{R}$ defined in the interval $0 \le x \le 1$. Given the Green function, $g(x, x')$, satisfying the equation

$$\frac{d^2 g}{dx^2} + k^2 g = -\delta(x - x')$$

with boundary conditions $g'(0, x') = g'(1, x') = 0$ where $g'(x, x') = \frac{dg(x,x')}{dx}$, show that the general solution is

$$\Psi(x) = \int_0^1 g(x, x')\rho(x')dx' \ ,$$

for homogeneous boundary conditions $\Psi'(0) = \Psi'(1) = 0$. You may use that the Green function is symmetric in its arguments $g(x, x') = g(x', x)$.

(b) Show that the Green function defined above is given by

$$g(x, x') = \begin{cases} A\cos kx \ , & x < x' \\ B\cos k(1-x) \ , & x > x' \end{cases}$$

Determine A and B by demanding G be continuous at $x = x'$ and satisfies the *jump condition:* $\lim_{\epsilon\to 0} g'(x' + \epsilon) - g'(x' - \epsilon) = -1$.

(c) Consider the case $k = 0$. Find the coefficients c_n so that

$$g(x; x') = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\sin(n\pi x') \qquad (3)$$

is a Green function on the interval $0 \le x \le 1$ satisfying Dirichlet boundary conditions at $x' = 0$ and $x' = 1$.