UNIVERSITY OF DUBLIN

XMA2331/2

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics **Trinity Term 2010**
SF Theoretical Physics
SF Two Subject Mod

COURSE 2331/2332: EQUATIONS OF MATHEMATICAL PHYSICS I/II

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ATTEMPT SIX QUESTIONS:
THREE FROM SECTION A AND THREE FROM SECTION B

All questions carry equal marks.

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

SECTION A

1. Sketch the triangular wave function defined by $f(x) = |x|$ for $-1 \leq x \leq 1$ and $f(x+2) = f(x)$, $\forall x$.

   Determine the (real) Fourier series of this function.

   Use this result (with Parseval's theorem) to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$$

$$\begin{aligned} a_0 &= \frac{2}{2}\int_{-1}^{1} dx|x| \\ &= 1. \end{aligned}$$

$$\begin{aligned} a_n &= \int_{-1}^{1} dx|x|\cos(\pi n x) \\ &= 2 \in_0^1 x\cos(\pi n x) \\ &= \frac{2}{n^2\pi^2}\left(\cos(n\pi x) - 1\right) \\ &= -\frac{4}{n^2\pi^2}, \quad \text{n odd.} \end{aligned}$$

Also,

$$b_n = \int_{-1}^{1} |x|\sin(n\pi x) = 0$$

since the integrand is odd. So

$$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\cos(\pi n x)$$

Then, rearranging the above gives

$$\frac{\pi^2}{8} = \sum_{n=1, n \text{ odd}} \frac{1}{n^2}\cos(n\pi x) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$$

2. Evaluate the surface integral, $\int\int_S \mathbf{F}\cdot\mathbf{dS}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}$ and the surface $\mathbf{S}$ is the upper half of the sphere $x^2+y^2+z^2 = 9$ and the disk $x^2+y^2 \le 9$ in the plane $z = 0$, with positive orientation.

   The sphere is parameterised as $3\sin\theta\cos\phi\mathbf{i} + 3\sin\theta\sin\phi\mathbf{j} + 3\cos\theta$. Then

   $$\frac{\partial r}{\partial\theta} = 3\cos\theta\cos\phi\mathbf{i} + 3\cos\theta\sin\phi\mathbf{j} - 3\sin\theta\frac{\partial r}{\partial\phi} = -3\sin\theta\sin\phi\mathbf{i} + 3\sin\theta\cos\phi\mathbf{j} + 0$$

   so that the cross product is

   $$\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\phi} = 9(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \cos\theta\sin\theta).$$

   Then the surface integral for the sphere is

   $$\begin{aligned}\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\mathbf{F}\cdot\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\phi} &= 2\pi(27)\int_0^{\pi/2}(\sin^3\theta + \cos^5\theta\sin\theta) \\ &= 2\pi(27)\left[\frac{2}{3}+\frac{9}{2}\right] \\ &= 279\pi.\end{aligned}$$

   For the disk (at $z = 0$) $S = \int_0^3\int_0^{2\pi} r^2 = 18\pi$ and the total surface integral is the sum of the two parts.

3. Prove that the Fourier transform of an even function is even.

   Express as a Fourier integral, the function

$$f(x) = \begin{cases} \cos(x) & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases} .$$

Proof: Have $f(x) = f(-x)$ so substituting in the fourier transform

$$f(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx f(-x) e^{-ikx}$$

Now write $u = -x$ so $du = -dx$ and

$$\begin{aligned} \tilde{f}(k) &= \int_{u=\infty}^{u=-\infty} -du f(u) e^{-ik(-x)} \\ &= \int_{-\infty}^{\infty} du f(u) e^{-i(-k)x} \\ &= \tilde{f}(-k). \end{aligned}$$

Writing f as a Fourier integral $f(x) = \int_{-\infty}^{\infty} dk\ e^{ikx}\ \tilde{f}(k)$. We require the Fourier transform:

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\ e^{-ikx}\ f(x) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dk\ e^{-ikx}\ \frac{e^{ix} + e^{-ix}}{2} \\ &= \frac{1}{4\pi} \left(\frac{e^{i(1-k)x}}{i(1-k)} + \frac{e^{i(-1-k)x}}{i(-1-k)} \right) \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{1}{4\pi} \left[\frac{ie^{-ik\pi/2} + ie^{ik\pi/2}}{i(1-k)} + \frac{-ie^{-ik\pi/2} - ie^{ik\pi/2}}{i(-1-k)} \right] \\ &= \frac{1}{4\pi}\, 2\cos\left(\frac{k\pi}{2}\right) \left(\frac{1}{1-k} + \frac{1}{1+k} \right) = \frac{1}{\pi} \cos\left(\frac{k\pi}{2}\right) \frac{1}{1-k^2}. \end{aligned}$$

Therefore

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dk\ \cos\left(\frac{k\pi}{2}\right) \frac{e^{ikx}}{1-k^2}.$$

Remark: $\tilde{f}(k)$ is well behaved at $k = \pm 1$.

4. Consider the rectangular box, B defined by $0 \leq x \leq a$, $0 \leq\leq b$, $0 \leq z \leq c$, for $a, b, c \in \mathbb{R}$. Evaluate

$$I = \int\int\int_B (xy^2 + z^3) dV.$$

$$\begin{aligned}
I &= \int\int\int_B (xy^2 + z^3) dV \\
&= \int_0^c dz \int_0^b dy \int_0^a dx (xy^2 + z^3) \\
&= \int_0^c dz \int_0^b dy \left(\frac{x^2y^2}{2} + xz^3 \right) \Big|_{x=0}^{x=a} \\
&= \int_0^c dz \left(\frac{a^2y^3}{6} + ayz^3 \right) \Big|_{y=0}^{y=b} \\
&= \frac{a^2b^3z}{6} + \frac{abz^4}{4} \Big|_{z=0}^{z=c} \\
&= \frac{a^2b^3c}{6} + \frac{abc^4}{4}.
\end{aligned}$$

Consider the tetrahedron T which has vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$. Evaluate

$$I = \int\int\int_T y dV.$$

$$\begin{aligned}
I &= \int\int\int_T y dV = \int dx \int_0^{1-x} dy \int_0^{1-x-y} dz y \\
&= \int dx \int_0^{1-x} dy y (1 - x - y) \\
&= \int dx \left((1-x) \frac{y^2}{2} - \frac{y^3}{3} \right)_0^{1-x} \\
&= \int_0^1 dx \frac{1}{6} (1 - x^3) = \frac{1}{24}.
\end{aligned}$$

SECTION B

5. State Stokes' theorem.
from notes

Use Stokes' theorem to evaluate $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}$ where the vector field $\mathbf{F} = z^2\mathbf{i} - 3xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the surface described by $z = 5 - x^2 - y^2$ above the $z = 1$ plane and oriented upwards.

Explain why Stokes theorem cannot by used to transform the surface (flux) integral

$$\int_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{dS}$$

into a line integral on the boundary of S.

Calculate $\oint \mathbf{F} \cdot \mathbf{dl}$ where the closed path is the circle $x^2 + y^2 = 4$, a boundary of the surface S at $z = 1$. Parameterise the path with $x = \cos u$, $y = \sin u$, $z = 1$ and $0 \le u \le 2\pi$. Then,

$$\frac{dr}{du} = (-\sin u, \cos u, 0)$$

and $\mathbf{F} \cdot dr/du = -\sin u(1 + 3\cos^2 u)$. The line integral is then

$$\oint \mathbf{F} \cdot \mathbf{dl} = \int_0^{2\pi} du(-\sin u - 3\cos^2 u \sin u) = 0.$$

For the second part note that if $\mathbf{F} = (x, y, z)$ then $div\mathbf{F} = 3 \neq 0$ and since $div(curl)\mathbf{F} = 0$ by definition then $\mathbf{F} = (x, y, z)$ cannot be a curl, so can't use Stokes.

6. Using the Frobenius method, find the general solution fo the ODE

$$4xy''(x) + 2y'(x) + y(x) = 0$$

where, as usual, the prime denotes differentiation with respect to x.

Comment on why the "naive" series solution approach will not work.

Write $y = \sum_{n=0}^{\infty} a_n\ x^{n+s}$ $y' = \sum_{n=0}^{\infty} a_n\ (n+s)\ x^{n+s}$, $\quad y'' = \sum_{n=0}^{\infty} a_n\ (n+s)(n+s-1)\ x^{n+s}$. Now: $y'(x) = a_0 s x^{s-1} + \sum_{n=0}^{\infty} a_{n+1}(n+1+s)x^{n+s}$ $xy''(x) = a_0 s(s-1)x^{s-1} + \sum_{m=0}^{\infty} a_{m+1}(m+1+s)(m+s)x^{m+s}$.

$$\begin{aligned} 4xy'' + 2y' + y &= a_0\left[4s(s-1) + 2s\right] \\ &+ \sum_{m=0}^{\infty}\left[4(m+1+s)(m+s)a_{m+1} + 2(m+1+s)a_{m+1} + a_m\right]x^{m+s} \\ &4a_0 s(s-\frac{1}{2})x^{s-1} + \sum_{m=0}^{\infty}\left[4(m+1+s)(m+s+\frac{1}{2})a_{m+1} + a_m\right]. \end{aligned}$$

Set $a_0 = 1$ Indicial equation: $s(s - \frac{1}{2}) = 0$ with roots $s = 0$ and $s = \frac{1}{2}$.

7. Use the Gauss (divergence) theorem to calculate $\int\int_S \mathbf{F}\cdot d\mathbf{S}$, where S is the surface of the box B with vertices $(\pm 1, \pm 2, \pm 3)$ with outward pointing normals and $\mathbf{F} = (x^2z^3, 2xyz^3, xz^4)$.

   Verify, by direct calculation, the divergence theorem for the case where $\mathbf{F} = (x, y, z)$ on the surface of the solid sphere of radius R centred at the origin.

8. Use separation of variables to solve the two-dimensional Laplace equation for $\Phi(x, y)$ with boundary conditions given by

$$\begin{aligned} \Phi(0, y) &= 0, \\ \frac{\partial}{\partial x}\Phi(0, y) &= \frac{1}{n}\sin(ny), \quad (n \in \mathbb{Z}). \end{aligned}$$

   called the Hadamard conditions.

   $\Phi(x, y) = X(x)Y(y)$ gives $X''Y + XY'' = 0$ or $\frac{X''}{X} = -\frac{Y''}{Y}$. Then

$$\begin{aligned} X'' &= EX \\ Y'' &= -EY \end{aligned} \tag{1}$$

   which each have solutions in the three classes as in the notes.

   Now apply the boundary conditions,

$$\begin{aligned} \Phi(0, y) &= X(0)Y(y) = 0 \\ \Phi(0, y) &= X'(0)Y(y) = \frac{1}{n}\sin ny \end{aligned} \tag{2}$$

   so we need $X(0) = 0$ and $X'(0) =$ constant. If this holds then we have

$$Y = \frac{A}{n}\sin ny \tag{3}$$

   so $E = n^2$ and hence

$$X = B\sinh nx \tag{4}$$

   where I have substituted

$$X = C_1e^{nx} + C_2e^{-nx} \tag{5}$$

   into the conditions on X and put the two exponentials together to get the $\sinh nx$. Now, putting this back together we get

$$\Phi(x, y) = \frac{C}{n}\sinh nx \sin nx \tag{6}$$

1 Some useful formulae

1. A function with period l has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right),$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x), \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x) \cos\left(\frac{2\pi n x}{l}\right), \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x) \sin\left(\frac{2\pi n x}{l}\right). \end{aligned}$$

2. A function with period l has a Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2i\pi n x}{l}\right),$$

where

$$c_n = \frac{1}{l}\int_{-l/2}^{l/2} dx f(x) \exp\left(\frac{-2i\pi n x}{l}\right).$$

3. The Fourier integral representation (or Fourier transform) is

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx}, \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx}. \end{aligned}$$