UNIVERSITY OF DUBLIN

XMA1231

TRINITY COLLEGE

Faculty of Science

school of mathematics

SF Mathematics
SF Theoretical Physics
SF Two Subject Mod

Trinity Term 2009

Course 231

Day Date, 2009

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ATTEMPT SIX QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. Show that the Jacobian, J, for the transformation from Cartesian to cylindrical polar coordinates, $dxdydz = Jdrd\phi dz$ is

$$J = r.$$

From notes - just note that this should be done for the 3-d case.

Find the volume of the region, R, lying below the plane $z = 3 - 2y$ and above the paraboloid, $z = x^2 + y^2$. Surfaces intersect at $x^2 + y^2 = 3 - 2y \Rightarrow x^2 + (y+1)^2 = 4$.

Now, dz ranges from $x^2 + y^2$ to $(3 - 2y)$. For dx, consider

$$\begin{aligned} x^2 &= 4 - (y+1)^2 \\ &= 4 - y^2 - 2y - 1 \\ \Rightarrow x &= \pm\sqrt{3 - y^2 - 2y} \end{aligned}$$

For dy, consider $x - 0$ so

$$\begin{aligned} y^2 + 2y + 1 &= 4 \\ (y+3)(y-1) &= 0 \end{aligned}$$

Then

$$\begin{aligned} V &= \int_{-3}^{1} dy \int_{-\sqrt{3-y^2-2y}}^{\sqrt{3-y^2-2y}} dx \int_{x^2+y^2}^{3-2y} dz \\ &= \int_{-3}^{1} dy \int_{-\sqrt{3-y^2-2y}}^{\sqrt{3-y^2-2y}} dx [3 - 2y - (x^2 + y^2)] \end{aligned}$$

Now the rest of this is messy so better to notice that $x^2 + (y+1)^2 = 4$ describes a disk D, radius 2, centre (0,-1) in the xy-plane. Then

$$\begin{aligned} V &= \int_D dxdy(3 - 2y - x^2 - y^2) \\ &= \int_D 4 - x^2 - (y+1)^2 \\ &= \int_0^{2\pi} \int_0^2 (4 - r^2) r dr \\ &= 2\pi(2r^2 - r^4/4)_0^2 \\ &= 8\pi \end{aligned}$$

changing variables to polars for the integration over x, y.

2. Show that for a conservative vector field, $\mathbf{F}$

$$\oint_C \mathbf{F} \cdot \mathbf{dl} = 0,$$

where C is any closed curve in the domain where $\mathbf{F}$ is defined. See notes.

Consider the vector field

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

Show that in the domain, $D = \{(x, y, z)|(x, y) \neq (0, 0)\}$, which is not simply-connected, the vector field $\mathbf{F}$ is not conservative.

Note that it is not sufficient to check $\nabla \times F = 0$ since the domain is **not** simply-connected. You must show that $\oint_C \mathbf{F} \cdot \mathbf{dl} = 0$.

A suitable closed curve is the circle parameterised by $x = \cos t, y = \sin t, z = 0$ and $0 \leq t \leq 2\pi$ so the line integral is

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{dl} &= \int_0^{2\pi} F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \\
&= \int_0^{2\pi} \sin^2 t + \cos^2 t \\
&= 2\pi
\end{aligned}$$

which is non-zero so the field is **not** conservative.

3. Consider the function,

$$g(x) = \begin{cases} \frac{-(\pi+x)}{2}, & -\pi \le x \le -1 \\ \frac{x(\pi-1)}{2}, & -1 < x < 1 \\ \frac{(\pi-x)}{2}, & 1 \le x \le \pi \end{cases} .$$

Verify that this is an odd function. Just show $g(-x) = -g(x)$ eg $g(-x) = -(\pi - x)/2 = (-\pi + x)/2 = -(\pi - x)/2 = -g(x)$ etc.

Determine the (real) Fourier series for the function $g(x)$. $g(x)$ odd means $a_n = 0 \forall n$ so just need the b_n for the Fourier series.

$$\begin{aligned} b_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x)\sin(2\pi nx/l) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} dx g(x)\sin(nx) \\ &= \frac{1}{\pi}\int_{-\pi}^{1} -\frac{\pi+x}{2}\sin(nx) + \frac{1}{\pi}\int_{-1}^{1}\frac{x(\pi-1)}{2}\sin(nx)\frac{1}{\pi}\int_{1}^{\pi}\frac{\pi-x)}{2}\sin(nx) \\ &= \frac{2}{\pi}\int_{0}^{1}\frac{x(\pi-1)}{2}\sin(nx) + +\frac{2}{\pi}\int_{1}^{\pi}\frac{\pi-x}{2}\sin(nx) \quad \dagger \\ &= \frac{\sin(n)}{n^2} \end{aligned}$$

where the first term in † is

$$\frac{\pi-1}{\pi}\int_0^1 x\sin(nx) = \frac{\pi-1}{\pi}\left[-\frac{1}{n}\cos(n) + \frac{1}{n^2}\sin(n)\right]$$

and the second term is

$$\frac{2}{\pi}\int_1^{\pi}\sin(nx) = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{1}{n}\cos(nx)_1^{\pi} - \frac{1}{2}\left[-\frac{x}{n}\cos(nx)_1^{\pi} + \frac{1}{n^2}\sin(nx)_1^{\pi}\right]\right]$$

Then

$$g(x) = \sum_0^{\infty}\frac{\sin(n)}{n^2}\sin(nx).$$

Use Parseval's theorem to show that

$$\sum_{n>0}^{n=\infty}\frac{\sin^2 n}{n^4} = \frac{(\pi-1)^2}{6}.$$

Theorem is

$$\frac{2}{l}\int_{-l/2}^{l/2}|f(x)|^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum(a_n^2 + b_n^2).$$

The a_n are zero so the RHS is $\sum_{n>0} \frac{\sin^2(n)}{n^4}$. The LHS is

$$\frac{1}{2\pi}\int_{-\pi}^{1} dx(pi^2 + 2\pi x + x^2) + \int_{-1}^{1} dx(x^2\pi^2 - 2x^2\pi + x^2) + \int_{1}^{\pi} dx(pi^2 - 2\pi x + x^2)$$

which is $(\pi - 1)^2/6$ after some algebra.

4. Prove that for a vector field $\mathbf{F}$,

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

See notes - note that one component is enough with a statement that the others follow.

State in full the Gauss (divergence) theorem.

See notes.

Consider S the surface described by the top half of the unit sphere, in $\mathbb{R}^3$, centred at the origin and oriented with upward pointing normals. Compute, using Gauss' theorem or otherwise, the flux of the vector field $\mathbf{F} = (xz^2+xy^2)\mathbf{i}+e^{x^2}\mathbf{j}+(x^2z+y^2)\mathbf{k}$ through S.

Gauss' theorem says the surface considered must be closed so use

$$\int_S F \cdot dS + \int_{S'} F \cdot dS = \int_D \nabla \cdot F dV$$

where S is the surface described and S' is the unit disk in the xy plane at $z = 0$ that closes it.

Now, $\nabla \cdot F = z^2 + y^2 + x^2 = r^2$ so in polars

$$\begin{aligned} \int_D dV divF &= \int_0^{2\pi} d\phi \int_0^1 dr \int_0^{\pi/2} d\theta r^2(r^2 \sin\theta) \\ &= \frac{2\pi}{5} \end{aligned}$$

For S' use the parameterisation $S'(r,\theta) = (r\cos\theta, r\sin\theta, 0)$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

For downward pointing normals the right choice is

$$\frac{\partial S'}{\partial \theta} \times \frac{\partial S'}{\partial r} = (0, 0, -r)$$

Then

$$\begin{aligned} \int_{S'} F dS &= \int_0^{2\pi} d\theta \int_0^1 dr (F_1, F_2, F_3) \cdot (0, 0, -r) \\ &= -\frac{1}{8} \end{aligned}$$

So putting it all together

$$\int F \cdot dS = \frac{2\pi}{5} + \frac{1}{8} = \frac{1}{40}(16\pi + 5).$$

5. Compute

(a) $\int_{-\infty}^{\infty} dx\delta(2x-3)\sin(\pi x)$

(b) $\int_{-\infty}^{\infty} dx e^{-|x|}\delta'(x-1)$

(c) $\int_{-5}^{5} dx\delta(x^2-3x+2)$

(d) $\frac{d}{dx}e^{a\theta(x)}$, a a constant and $\theta(x)$ the Heaviside function.

6. Using the Frobenius method or otherwise find the general solution of the ODE,

$$4xy''(x)+2y'(x)+y(x)=0,$$

where as usual the prime denotes differentiation with respect to x.

7. Given a function, $\Phi(x,y)$ which is harmonic in the square described by $0\leq x\leq\pi$ and $0\leq y\leq\pi$. Suppose that Φ is zero on three sides of this square (ie $\Phi(0,y)=\Phi(\pi,y)=\Phi(x,\pi)=0$) and on the lower side

$$\Phi(x,0)=\cos(x).$$

Determine $\Phi(x,y)$ within the square.

Note that this question was covered in lectures and is also on a problem set.

8. Determine the general solution to Hermite's equation

$$y''-2xy'+2\alpha y=0.$$

For $\alpha=2$ and $\alpha=3$ determine the corresponding Hermite polynomial solutions.

The first bit was done in lectures, giving

$$a_{n+2}=\frac{2(n-\alpha)}{(n+1)(n+2)}a_n$$

giving $y=C_1y_{even}(x)+C_2y_{odd}(x)$ for even and odd recursion relations.

Now for $\alpha=2$, $a_{n+2}=2(n-2)a_n/(n+1)(n+2)$ and considering n even and $a_0=1$ gives: for $n=0, a_2=-2$, for $n=2, a_4=0$ and so $a_6=\ldots=0$. Then

$$y_e=\sum a_nx^n=a_0x^0+a_2x^2=1-2x^2.$$

For $\alpha=3$, $a_{n+2}=2(n-3)a_n/(n+1)(n+2)$. Consider n odd and $a_1=1$ gives: for $n=1, a_3=-2/3$ and for $n=3, a_5=0$ and so $a_7=\ldots=0$. Then

$$y_o=\sum a_nx^n=a_1x^1+a_3x^3=x-\frac{2}{3}x^3.$$

1 Some useful formulae

1. A function with period l has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{l}\right),$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x), \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x) \cos\left(\frac{2\pi nx}{l}\right), \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x) \sin\left(\frac{2\pi nx}{l}\right). \end{aligned}$$

2. A function with period l has a Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2i\pi nx}{l}\right),$$

where

$$c_n = \frac{1}{l}\int_{-l/2}^{l/2} dx f(x) \exp\left(\frac{-2i\pi nx}{l}\right).$$

3. The Fourier integral representation (or Fourier transform) is

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx}, \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx}. \end{aligned}$$