UNIVERSITY OF DUBLIN

XMA2331/2

TRINITY COLLEGE

Faculty of Science

school of mathematics

SF Mathematics **Michaelmas Term 2011**
SF Theoretical Physics
SF Two Subject Mod

Course 2331/2332: Equations of Mathematical Physics I/II

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ATTEMPT SIX QUESTIONS:
THREE FROM SECTION A AND THREE FROM SECTION B

All questions carry equal marks.

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

SECTION A

1. Show that the periodic function f defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$ has the Fourier series expansion

$$f(x) = -\frac{4}{\pi} \sum_{n>0,\ \text{odd}} \frac{\cos nx}{n^2}.$$

Use the Fourier series given above to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + ... \quad \text{and} \quad S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ...$$

2. Prove that

$$\int_{-\infty}^{\infty} \delta(h(x)) f(x) = \sum_{i=1}^{n} \frac{f(x_i)}{|h'(x_i)|}.$$

where $h(x)$ is a smooth function with roots $x = x_i$.

Compute

(a) $\int_{-\infty}^{\infty} dx \delta(1 + x^2) f(x)$

(b) $\int_{-\infty}^{\infty} dx \delta(x^2 + x)$

(c) $\int_0^2 dx e^x \delta'(x - 1)$

(d) $\int_0^1 dx \delta\left(\frac{1}{\sin x}\right)$

3. Compute

(a) the line integral of the vector field $\mathbf{F}(x) = (e^x, xy^2)$ along the boundary of the triangle with vertices $(0, 0), (1, 0)$ and $(1, 1)$. Move along the boundary in the positive x-direction from $(0, 0)$ to $(1, 0)$.

(b) the line integral of the vector field $\mathbf{F}(x) = (yx^2 + z^3, z^2, 2yz)$ along the positively oriented intersection of the cyliner $x^2 + y^2 = 1$ and the plane $z = x + 3$.

4. Prove that the Fourier transform of an even function is even.

Express as a Fourier integral, the function

$$f(x) = \begin{cases} \cos(x) & |x| < \frac{\pi}{2} \\ 0 & |x| > \frac{\pi}{2} \end{cases} .$$

SECTION B

5. Consider the function $\phi(x, y)$ which is harmonic in the square defined by $0 \leq x \leq \pi, 0 \leq y \leq \pi$. ϕ is zero on three sides of the square and on the fourth side $\phi(x, 0) = \sin x$.

   Determine $\phi(x, y)$ within the square.

6. Consider the Bessel equation, given by

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0,$$

   where as usual, the prime denotes differentiation with respect to x.

   Use the Frobenius method to show that a recursion relation for the coefficients in a power series solution is given by

$$a_{n+2} = \frac{a_n}{(n \pm \nu + 2)^2 - \nu^2}.$$

   For the choice $\nu = \frac{1}{2}$ determine the recursion relations and thence show that the general solution is

$$y(x) = x^{-\frac{1}{2}} \left(C_1 \cos x + C_2 \sin x \right).$$

7. State in full the Gauss (divergence) theorem.

   Consider S the surface described by the top half of the unit sphere, in $\mathbb{R}^3$, centred at the origin and oriented with upward pointing normals. Compute, using Gauss' theorem or otherwise, the flux of the vector field $\mathbf{F} = (xz^2 + xy^2)\mathbf{i} + e^{x^2}\mathbf{j} + (x^2 z + y^2)\mathbf{k}$ through S.

8. Using Stoke's Theorem, calculate $\int_S \mathbf{F} \cdot \mathbf{dS}$ where $\mathbf{F} = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$ and S is the paraboloid described by $z = 9 - x^2 - y^2$ and oriented upwards with $z > 0$.

   Use Stoke's theorem to show that

$$\int_S f(\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_S (\mathbf{A} \times \nabla f) \cdot d\mathbf{S} + \oint_C f\mathbf{A} \cdot d\mathbf{l}$$

   where S is an open surface and C is its perimeter.

1 Some useful formulae

1. A function with period l has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right),$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x), \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x)\cos\left(\frac{2\pi n x}{l}\right), \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x)\sin\left(\frac{2\pi n x}{l}\right). \end{aligned}$$

2. A function with period l has a Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2i\pi n x}{l}\right),$$

where

$$c_n = \frac{1}{l}\int_{-l/2}^{l/2} dx f(x)\exp\left(\frac{-2i\pi n x}{l}\right).$$

3. The Fourier integral representation (or Fourier transform) is

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx}, \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx}. \end{aligned}$$