UNIVERSITY OF DUBLIN

XMA1231

TRINITY COLLEGE

Faculty of Science

school of mathematics

SF Mathematics
SF Theoretical Physics
SF Two Subject Mod
Supplemental Examination

Michaelmas Term 2009

Course 231

Day Date, 2009

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ATTEMPT SIX QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. In two dimensions, write down the change of variables from Cartesian to polar coordinates. Calculate the appropriate Jacobian for this transformation.

   Consider the half-disk, $x^2 + y^2 \leq 1$ and $x > 0$ enclosing a region, D of the xy-plane. Sketch this region and evaluate its area.

   The centres of mass are defined as

$$\bar{x} = \frac{\int_D dA\rho(x,y)x}{\int_D dA\rho(x,y)}, \quad \bar{y} = \frac{\int_D dA\rho(x,y)y}{\int_D dA\rho(x,y)}.$$

   where $\rho(x,y)$ is the density. For uniform density ie. $\rho(x,y) = 1$ determine the centres of mass.

2. Show that if a continuous vector field $\mathbf{F}$ in an open and connected domain D is conservative then it is path-independent.

   Compute the following line integrals.

   (a) $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle in the xy-plane, taken anticlockwise and $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.

   (b) $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is again the unit circle in the xy-plane, taken anticlockwise and $\mathbf{F} = y\mathbf{i} - x^2 y\mathbf{j}$.

   (c) $\int_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle in the first quadrant ie $x^2 + y^2 = 1, x \geq 0, y \geq 0$, taken anticlockwise and $\mathbf{F} = x\mathbf{j}$.

3. Consider the function

$$f(x) = x^2, \text{ periodic on the interval } [-\pi, \pi].$$

   Determine the (real) Fourier series for $f(x)$.

   Use this result and Parseval's identity to determine the following closed-form expression for the Riemann zeta function at $s = 4$, namely that

$$\zeta(4) = \frac{\pi^4}{90}, \quad \text{noting that} \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \ (s > 1).$$

4. Given a function, $\Phi(x,y)$ which is harmonic in the square described by $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$. Suppose that Φ is zero on three sides of this square (ie $\Phi(0,y) = \Phi(\pi,y) = \Phi(x,\pi) = 0$) and on the lower side

$$\Phi(x,0) = \sin(x).$$

   Determine $\Phi(x,y)$ within the square.

5. State Stokes' theorem.

   Use Stokes' theorem to evaluate $\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}$ where the vector field $\mathbf{F} = z^2\mathbf{i} - 3xy\mathbf{j} + x^3y^3\mathbf{k}$ and S is the surface described by $z = 5 - x^2 - y^2$ above the $z = 1$ plane and oriented upwards.

   Explain why Stokes theorem cannot by used to transform the surface (flux) integral

$$\int_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{dS}$$

   into a line integral on the boundary of S.

6. Prove that the Fourier transform of an even function is even.

   Compute the Fourier transform of $f(x) = e^{-a|x|}$ where a is a positive constant.

   Use the result to show that
$$\int_{-\infty}^{\infty} \frac{\cos(p)}{1+p^2} = \frac{\pi}{e}.$$

7. Using the Frobenius method find a series solution of the ODE,

$$x(x-1)y''(x) + 3xy'(x) + y(x) = 0,$$

   where as usual the prime denotes differentiation with respect to x.

   Comment on the behaviour of the second solution determined from the Frobenius method.

   Suggest a method to construct the second solution.

8. Prove that

   (a) $\displaystyle\int_{-\infty}^{\infty} dx \delta'(x) f(x) = -f'(0).$

   (b) $\displaystyle\int_{-\infty}^{\infty} dx \delta(h(x)) f(x) = \sum_{i=1}^{n} \frac{f(x_i)}{|h'(x_i)|}$, where x_i are the roots of the continuous function $h(x)$.

   Compute

   (i) $\displaystyle\int_{-\infty}^{\infty} dx \delta(x^2 - x - 2).$

   (ii) $\displaystyle\int_{0}^{1} dx \delta(\sin(1/x)).$

   (iii) $\displaystyle\frac{d}{dx} e^{a\theta(x)}$ where a is a constant and $\theta(x)$ is the Heaviside function (see page of useful formulae).

1 Some useful formulae

1. A function with period l has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right),$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x), \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x) \cos\left(\frac{2\pi n x}{l}\right), \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x) \sin\left(\frac{2\pi n x}{l}\right). \end{aligned}$$

2. A function with period l has a Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2i\pi n x}{l}\right),$$

where

$$c_n = \frac{1}{l}\int_{-l/2}^{l/2} dx f(x) \exp\left(\frac{-2i\pi n x}{l}\right).$$

3. The Fourier integral representation (or Fourier transform) is

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx}, \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx}. \end{aligned}$$

4. The Dirac delta function:

$$\int_{-\infty}^{\infty} dx f(x)\delta(x) = f(0).$$

5. The Heaviside function $\theta(x)$ is

$$\theta(x) = \left\{ \begin{array}{c} 1, x \geq 0 \\ 0, x < 0 \end{array} \right\}.$$