# UNIVERSITY OF DUBLIN

XMA1231

## TRINITY COLLEGE

Faculty of Science

school of mathematics

**SF Mathematics**
**SF Theoretical Physics**
**SF Two Subject Mod**

**Hilary Term 2009**

Course 231

Day Date, 2009

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ATTEMPT SIX QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. Express the function $f(x) = |cos(x)|$ as a (real) Fourier series.

   This has period $\pi$, so

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx|\cos(x)|\cos(nx) = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx\cos(x)\cos(nx).$$

   For $n = 0$,

$$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx\cos(x) = \frac{4}{\pi}.$$

   and

$$\begin{aligned} a_n &= \frac{1}{2}\frac{2}{\pi}\int_{-\pi/2}^{\pi/2} dx\left(e^{i(n+1)x} + e^{i(-n+1)x}\right), \\ &= \frac{1}{\pi}\frac{1}{i}\frac{e^{(n+1)ix}}{n+1} + \frac{e^{i(-n+1)x}}{-n+1}\Bigg|_{-\pi/2}^{\pi/2}, \\ &= \frac{2}{\pi}\left(\frac{1}{n+1}\sin\left((n+1)\frac{\pi}{2}\right) + \frac{1}{-n+1}\sin\left((1-n)\frac{\pi}{2}\right)\right). \end{aligned}$$

   For $n$ odd, $\sin\left((n\pm 1)\frac{\pi}{2}\right) = 0$. For $n$ even, $\sin\left((n+1)\frac{\pi}{2}\right) = \cos(n\pi/2) = (-1)^{n/2}$ and $\sin\left((-n+1)\frac{\pi}{2}\right) = \cos(n\pi/2) = (-1)^{n/2}$. So,

$$a_n = \frac{2}{\pi}\left(\frac{1}{n+1} + \frac{1}{-n+1}\right)(-1)^{n/2} = \frac{4}{1-n^2}.$$

   So, the fourier series is

$$|\cos(x)| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{1-4n^2}\cos(2nx).$$

   where $n \to 2n$ for n even.

2. Evaluate the integral

$$\int_D dV\sqrt{3x^2+3z^2},$$

where $D$ is the solid bounded by $y = 2x^2 + 2z^2$ and the plane $y = 8$.

You should begin by sketching the solid of interest.

Use spherical polars in the $xz$ plane. Write $x = r\cos\theta$ and $z = r\sin\theta$ so that $x^2 + z^2 = r^2$. Then $2x62 + 2z^2 \le y \le 8$ with $0 \le \theta \le 2\pi$ and $0 \le r \le 2$ (from $z = 0$ in $8 = 2x62 + 2z^2$). Then

$$\begin{aligned}\int_D dV\sqrt{3x^2+3z^2} &= \int_0^{2\pi} d\theta \int_0^2 dr \int_{2x62+2z^2}^8 dy\sqrt{3x^2+3z^2},\\ &= \int_0^{2\pi} d\theta \int_0^2 dr\ \sqrt{3x^2+3z^2}y\Big|_{2x62+2z^2}^8,\\ &= \int_0^{2\pi} d\theta \int_0^2 dr\sqrt{3x^2+3z^2}(8-(2x^2+2z^2)).\end{aligned}$$

Now $x^2 + z^2 = r^2$ so we can rewrite the integral above as

$$\begin{aligned}\int_D dV\sqrt{3x^2+3z^2} &= \int_0^{2\pi} d\theta \int_0^2 dr\sqrt{3r62}(8-2r^2)r,\\ &= 2\pi\int_0 62dr\sqrt{3}(8r^2-2r^4),\\ &= \sqrt{3}2\pi\left[\frac{8}{3}r^3-\frac{2}{5}r^5\right]_0^2,\\ &= 2\pi\sqrt{3}\left[\frac{64}{3}-\frac{64}{5}\right]\\ &= \frac{256\sqrt{3}\pi}{15}.\end{aligned}$$

3. An *eigenfunction* of the Fourier transform $(\widetilde{f(k)})$, is a function $f$ for which

$$\widetilde{f(k)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx} = \lambda f(k),$$

where $\lambda$ is a scalar called the *eigenvalue.*

Show that the Gaussian function $f(x) = e^{-\frac{1}{2}x^2}$ is such an eigenfunction.

*Note: The gaussian integral formula is given in the useful information page and may be used without proof*

Consider the function

$$\Phi(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_t(x),$$

the so-called generating function for the Hermite polynomials $H_t$. By computing the fourier transform of $e^{-\frac{1}{2}x^2}\Phi(x,t)$ show that the functions $f_n(x) = e^{-\frac{1}{2}x^2} H_t(x)$ are eigenfunctions of the Fourier transform.

Determine the eigenvalues $\lambda_n$.

Work out $\widetilde{f(k)}$ by

$$\begin{aligned}
\widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} e^{-ikx} \\
&= \frac{1}{2\pi}\sqrt{\frac{\pi}{\frac{1}{2}}} e^{-ik^2/4(\frac{1}{2})} \\
&= \frac{1}{2\pi}\sqrt{2\pi} e^{-k^2/2} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}k^2} \\
&= \frac{1}{\sqrt{2\pi}} f(k).
\end{aligned}$$

Now, $\Phi(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_t(x)$. Taking the FT of $e^{-\frac{1}{2}x^2}\Phi(x,t)$,

$$\begin{aligned}
\mathcal{F}(e^{-\frac{1}{2}x^2}\Phi(x,t)) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2+2xt-t^2-ikx} \\
&= e^{-\frac{k^2}{2}-2kit+t^2} \\
&= \sum_{0}^{\infty} e^{-\frac{k^2}{2}} H_n(k)\frac{(-it)^n}{n!}.
\end{aligned}$$

And the FT of $e^{-\frac{1}{2}x^2}\sum_{n=0}^{\infty}\frac{t^n}{n!}H_t(x)$ can be written as $\sum_{n=0}^{\infty}\mathcal{F}(e^{-x^2/2}H_n(x))\frac{t^n}{n!}$. Equating like powers of $n$ on the left and right sides of the FT gives

$$\mathcal{F}(e^{-x^2/2}H_n(x)) = (-i)^n e^{-k^2/2} H_n(k).$$

ie eigenfunctions as required and using the eigenfunction result above the eigenvectors are $(-i)^n$.

4. Prove that a continuous vector field $\mathbf{F}$ in an open and connected domain, is conservative if and only if it is path independent.

   Given the vector field, $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$, determine the line integral $\int_C \mathbf{F} \cdot d\mathbf{l}$ from $(1, 0)$ to $(0, -1)$ along

   - the straight line segment joining these points,
   - three-quarters of the unit circle, centred at the origin traversed counterclockwise.

   What do the results imply about the field $\mathbf{F}$?

   Proof in the notes.

   - the straight line segment:
     Parameterise the curve as $\mathbf{r} = (1-t)\mathbf{i} - t\mathbf{y}$, with $0 \leq t \leq 1$. So, $\partial \mathbf{r}/\partial t = -\mathbf{i} - \mathbf{y}$ and also $\mathbf{F} = -t\mathbf{i} + (t-1)\mathbf{y}$ . Then the line integral is

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{dl} &= \int_0^1 dt\, ((-t)\mathbf{i} + (t-1)\mathbf{y}) \cdot (-\mathbf{i} - \mathbf{y}) \\ &= \int_0^1 dt t + 1 - t \\ &= 1. \end{aligned}$$

   - This time parameterise the curve as $\mathbf{r} = \cos(t)\mathbf{i} + \sin(t)\mathbf{y}$ giving $\partial \mathbf{r}/\partial t = -\sin(t)\mathbf{i} + \cos(t)\mathbf{y}$ and $\mathbf{F} = \sin(t)\mathbf{i} - \cos(t)\mathbf{y}$. The line integral is

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{dl} &= \int_0^{3\pi/2} (\sin(t)\mathbf{i} - \cos(t)\mathbf{y}) \cdot (-\sin(t)\mathbf{i} + \cos(t)\mathbf{y}) \\ &= \int_0^{3\pi/2} -1 \\ &= -\frac{3\pi}{2}. \end{aligned}$$

   The line integrals along the 2 paths differ so $\mathbf{F}$ is path-dependent and not conservative.

# 1 Some useful formulae

1. A function with period $l$ has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right),$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x), \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x) \cos\left(\frac{2\pi n x}{l}\right), \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx f(x) \sin\left(\frac{2\pi n x}{l}\right). \end{aligned}$$

2. A function with period $l$ has a Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2i\pi n x}{l}\right),$$

where

$$c_n = \frac{1}{l}\int_{-l/2}^{l/2} dx f(x) \exp\left(\frac{-2i\pi n x}{l}\right).$$

3. The Fourier integral representation (or Fourier transform) is

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx}, \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx f(x) e^{-ikx}. \end{aligned}$$

4. The Gaussian integral is

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a},$$

for $a > 0$ and $b \in \mathbb{C}$.