Q1

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{Vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{Vmatrix}$$

$$= \begin{vmatrix} s\theta c\phi & rc\theta c\phi & -rs\theta s\phi \\ s\theta s\phi & rc\theta s\phi & rs\theta c\phi \\ c\theta & -rs\theta & 0 \end{vmatrix}$$

$= r^2 \sin\theta$ $\longleftarrow$ show workings.

$x^2+y^2+z^2 = 2$, convert to sphericals $\Rightarrow r = \sqrt{2}$

For the cone $z^2 = x^2+y^2$

$\Rightarrow 2z^2 = x^2+y^2+z^2$

$\Rightarrow 2r^2\cos^2\phi = r^2$

$\Rightarrow \phi = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ or $\frac{7\pi}{4}$.

and $7\pi/4 = 3\pi/4$ (sphericals $\rightarrow \pi$ periodicity)

So $V = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^{\sqrt{2}} r^2 \sin\phi \, dr \, d\phi \, d\theta$

$= \frac{8\pi}{3}$

Q2.

Proof from notes.

For $\underline{F} = (3x^2 + 6y)\underline{i} - 14yz\underline{j} + 20xz^2\underline{k}$.

a) $r(t) = t\underline{i} + t^2\underline{j} + t^3\underline{k}$ (parameterised path).

$\frac{dr}{dt} = \underline{i} + 2t\underline{j} + 3t^2\underline{k}$

$\underline{F}(t) = 9t^2\underline{i} - 14t^5\underline{j} + 20t^7\underline{k}$

$\Rightarrow \underline{F} \cdot \frac{dr}{dt} = 9t^2 - 28t^6 + 60t^9$

so $\int \underline{F} \cdot dl = \int_0^1 dt\ 9t^2 - 28t^6 + 60t^9 = 5.$

b) $r(t) = t\underline{i} + t\underline{j} + t\underline{k}$, $\frac{dr}{dt} = \underline{i} + \underline{j} + \underline{k}$

so $\underline{F} \cdot \frac{dr}{dt} = 6t - 11t^2 + 20t^3$

so $\int \underline{F} \cdot dl = \int_0^1 dt\ 6t - 11t^2 + 20t^3 = 3 - \frac{11}{3} + 5.$

ii) $\cos \omega t = \frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t}$

$$F(s) = \int_0^\infty e^{-st}\left(\tfrac{1}{2} e^{i\omega t}\right) + \tfrac{1}{2} e^{-st}\left(e^{-i\omega t}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{(-s+i\omega)t} dt + \frac{1}{2}\int_0^\infty e^{(-s-i\omega)t} dt$$

$$= \frac{1}{2}\frac{1}{(s-i\omega)} + \frac{1}{2}\frac{1}{s+i\omega}$$

$$= \frac{s}{s^2+\omega^2}$$

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i) $e^{\omega t}$.

$$F(s) = \int_0^\infty e^{\omega t} e^{-st} dt$$

$$= \int_0^\infty e^{(\omega - s)t} dt$$

$$= \frac{1}{\omega - s} e^{(\omega - s)t} \Big|_0^\infty = \cancel{\frac{1}{\omega - s}} = \frac{1}{s-\omega}.$$

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Now

$$\mathcal{L}(f') = \int_0^\infty f'(t) e^{-st} dt$$

$$= f(t) e^{-st} \Big|_0^\infty + s\int_0^\infty f(t) e^{-st} dt$$

$$= -f(0) + s\mathcal{L}(f(t))$$

$$= sF(s) - f(0).$$

~~3~~

Now

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$$f' - f = 0$$

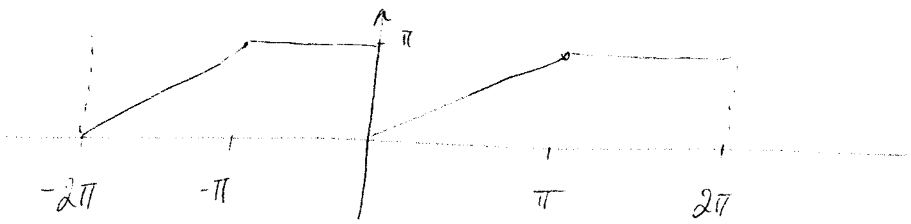
$$\mathcal{L}(f') - \mathcal{L}(f) = 0$$

$$sF(s) - f(0) - \mathcal{L}(f) = 0 \Rightarrow s\mathcal{L}(f) - \mathcal{L}(f) - f(0) = 0$$

$$(s-1)\mathcal{L}(f) = 1$$

$$\Uparrow \quad \mathcal{L}(f) = \frac{1}{s-1} = e^t$$

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi \end{cases} \qquad f(x) = f(x+2\pi).$$ (4)



$$a_0 = \frac{2}{l}\int_{-l/2}^{l/2} dx\, f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\,dx$$
$$= \frac{1}{\pi}\,\frac{1}{2}x^2\Big|_0^{\pi} + \frac{\pi}{\pi}\,x\Big|_{\pi}^{2\pi}$$
$$= \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) + \pi$$
$$= \frac{3\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cos nx\,dx$$
$$= \frac{1}{\pi}\left[x\,\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right] + \frac{\pi}{\pi}\,\frac{\sin nx}{n}\Big|_{\pi}^{2\pi}$$
$$= \frac{1}{\pi}\;\frac{1}{n}(\pi\sin n\pi - 0) - \left(\frac{-\cos nx}{n^2}\Big|_0^{\pi}\right) + \frac{1}{n}(\sin n2\pi - \sin n\pi)$$
$$= \frac{1}{\pi}\left[\frac{1}{n}(0)\right) + \frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\Big] + \frac{1}{n}(0-0)$$
$$= \frac{1}{n^2\pi}\,(-1)^n - 1$$

$$a_n = \begin{cases} -\dfrac{2}{n^2\pi}, & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x \sin nx \, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)_0^{\pi} - \int_0^{\pi} -\frac{\cos nx}{n}\, dx \right] + \frac{\pi}{\pi}\left(-\frac{\cos nx}{n}\right)_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi \cos n\pi}{n} + 0\right) + \frac{\sin n\pi}{n^2}\Big|_0^{\pi}\right] - \frac{1}{n}\left(\cos 2n\pi - \cos n\pi\right)$$

$\frac{1}{2}$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \frac{\sin n\pi - \sin 0}{n^2}\right] - \frac{1}{n}\left(1 - (-1)^n\right)$$

$$= -\frac{1}{n}(-1)^n - \frac{1}{n}\left(1-(-1)^n\right)$$

$$= -\frac{1}{n}$$

$$\Rightarrow \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}{}' \left(a_n \cos nx + b_n \sin nx\right)$$

$$= \frac{3\pi}{4} + \sum_{n \text{ odd}}{}' -\frac{2}{n^2\pi}\cos nx + \sum_{n}{}' -\frac{1}{n}\sin nx$$

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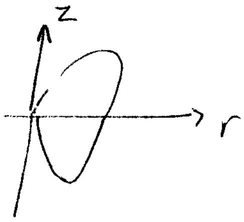
$r = \cos u$
$z = \sin 2u$ $\quad , -\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$\circlearrowleft z$ axis

from div thm $\int_C dv \ \operatorname{div} F = \int_S F \cdot dA$ (2)

Want $\iiint dV = \iint_S F \cdot dA$

where F is a vec. field of div 1, choose $F = z\underline{k}$ 3



$t = \theta$
$r = \cos u$
$z = \sin 2u$
$0 \le t \le 2\pi$

$\Rightarrow$ $x = \cos u \cos t$
$y = \cos u \sin t$
$z = \sin 2u$
$-\pi/2 \le u \le \frac{\pi}{2}.$

$$\Rightarrow d\underline{A} = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{vmatrix} = \begin{vmatrix} i & j & k \\ -cust & cuct & 0 \\ -suct & -sust & 2c2u \end{vmatrix} dt\,du$$ 2.

$$= (2\cos u \cos 2u \cos t,\ 2\cos u \cos 2u \sin t,\ \cos u \sin u)$$

$$\Rightarrow \int_S z\underline{k} \cdot dA = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} (\sin 2u \times \cos u \sin u)\, dt\, du$$

$$= 2\pi \int_{-\pi/2}^{\pi/2} \sin 2u \cos u \sin u \, du$$ 7

$$= 2\pi \left[\frac{1}{16}(4u - \sin 4u) \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi^2}{2}.$$

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$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{at} e^{-ikx}$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} dx\, e^{-ikx} e^{-at} + \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{at}$$

$$= \frac{1}{2\pi}\left[-\frac{e^{-x(a+ik)}}{a+ik}\Big|_{0}^{\infty} - \frac{e^{x(a-ik)}}{a-ik}\Big|_{-\infty}^{0} \right].$$

$$= \frac{1}{2\pi}\left[\frac{1}{a+ik} + \frac{1}{a-ik} \right]$$

$$= \frac{1}{\pi}\,\frac{a}{a^2+k^2}.$$

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then

$$f(x) = \int_{-\infty}^{\infty} dk\, e^{ikx}\tilde{f}(k) = \frac{a}{\pi}\int_{-\infty}^{\infty} dk\, \frac{e^{ikx}}{a^2+k^2}$$

3

$$\tilde{f}(\cos x) = \int_{-\infty}^{\infty} dx\, \cos x\, e^{-ikx}$$

0

$$= \frac{1}{2\pi}\frac{1}{2}\int_{-\infty}^{\infty} dx\left[e^{-i(k-1)x} + e^{-i(1+k)x} \right]$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-i(k-1)x} + e^{-i(1+k)x}$$

$$= \frac{1}{4\pi}\left[\delta(k-1) + \delta(k+1) \right]$$ by def^n of $\delta(x)$

$$F = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} \quad \text{has} \quad \nabla \cdot F = 0.$$ (7)

$$\nabla \cdot F = \frac{\partial}{\partial x}\frac{x}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} + \frac{\partial}{\partial z}\frac{z}{r^3}$$

eg
$$\frac{\partial x}{\partial x r^3} = \frac{r^3 - 3x(x/r)r^2}{r^6} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$ 6

$$\Rightarrow \nabla \cdot F = \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} = 0.$$ $\Big/ \frac{\partial}{\partial x} r = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{r}.$

Prove:

$$\nabla \cdot (\nabla \times F) = 0;$$

$$\nabla \times F = (\partial_y F_3 - \partial_z F_2)i + (\partial_z F_1 - \partial_x F_3)\underline{j} + (\partial_x F_2 - \partial_y F_1)\underline{k}.$$

$$\Rightarrow \nabla \cdot (\nabla \times F) = \partial_x(\partial_y F_3 - \partial_z F_2) + \partial_y(\partial_z F_1 - \partial_x F_3) + \partial_z(\partial_x F_2 - \partial_y F_1)$$
$$= \underset{=}{\partial_x\partial_y F_3} - \underset{\alpha}{\partial_x\partial_z F_2} + \underset{\#}{\partial_y\partial_z F_1} - \underset{=}{\partial_y\partial_x F_3} + \underset{\alpha}{\partial_z\partial_x F_2} - \underset{\#}{\partial_z\partial_y F_1}$$
$$= 0.$$

$$\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \Delta F$$

now
$$(\nabla \times (\nabla \times F))_1 = \partial_y(\partial_x F_2 - \partial_y F_1) - \partial_z(\partial_z F_1 - \partial_x F_3)$$
$$\cancel{\partial_y(F_{2,x} - F_{1,y}) - \partial_z(F_{3,x} - F_{1,z})}$$
$$= F_{2,xy} - F_{1,yy} + F_{3,xz} - F_{1,zz}$$ 8

other side.

$$(\nabla(\nabla \cdot F))_1 = \partial_x(F_{1,x} + F_{2,y} + F_{3,z}) = F_{1,xx} + F_{2,yx} + F_{3,zx}$$

$$\Rightarrow \nabla \times (\nabla \times F) - \nabla(\nabla \cdot F) = -F_{1,yy} - F_{1,xx} - F_{1,zz}$$
$$= -(\Delta F)_1$$

other compt. sim.

aux eq^n

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow \quad \lambda + 1)(\lambda + 2) = 0 \quad \Rightarrow \quad \lambda = -2, -1$$

2

RHS. as f.s.

$$f(x) = \frac{a}{\pi} + \sum_{n \neq 0} \frac{\sin an}{\pi n} e^{inx}$$

ie

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\, f(x) = \frac{1}{\pi}\int_{-a}^{a} dx = \frac{1}{\pi}(x)_{-a}^{a} = \frac{2a}{\pi}$$

2

$$\Rightarrow \frac{a_0}{2} = \frac{a}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx = \frac{1}{\pi}\int_{-a}^{a} \cos nx = \frac{1}{\pi} \frac{\sin nx}{n}\Big|_{-a}^{a}$$

$$= \frac{1}{n\pi} \sin an - \sin(-an)$$

$$= \frac{1}{n\pi} \sin an + \sin an$$

$$= \frac{2 \sin an}{n\pi}$$

$$b_n = \frac{1}{\pi}\int_{-a}^{a} \sin nx = \frac{1}{\pi} \frac{\cos nx}{n}\Big|_{-a}^{a} = \frac{1}{n\pi} \cos(ax) - \cos(-ax)$$

$$= \frac{1}{n}(\cos ax - \cos ax) = 0$$

$$\Rightarrow \quad f(x) = \frac{a}{\pi} + \sum_{\substack{n=0 \\ n \neq 0}}^{\infty} \frac{2 \sin an}{n\pi} \cos nx$$

2

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$$\to \frac{a}{\pi} \sum_{n, n \neq 0}' \frac{\sin an}{n\pi} e^{inx}$$

so solve $y'' + 3y' + 2y = \frac{\sin an}{\pi n} e^{inx}$

let $y = Ce^{inx} \Rightarrow (-n^2 + 3in + 2)C = \frac{\sin an}{\pi n}$ and for $n = 0$ $y = \frac{a}{2\pi}$

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$$\Rightarrow \quad y = C_1 e^{-x} + C_2 e^{-2x} + \frac{a}{2\pi} + \sum_{n, n \neq 0} \frac{1}{-n^2 + 3in + 2} \frac{\sin an}{\pi n} e^{inx}$$

2