UNIVERSITY OF DUBLIN

XMA1231

TRINITY COLLEGE

Faculty of Science

school of mathematics

SF Mathematics
SF Theoretical Physics
SF Two Subject Mod

Hilary Term 2009

Course 231

Day Date, 2009

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ATTEMPT SIX QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. Determine the Jacobian for the transformation from Cartesian to spherical polar coordinates in three dimensions.

   Find the volume of the solid that lies inside the sphere

$$x^2 + y^2 + z^2 = 2,$$

   and outside the cone

$$z^2 = x^2 + y^2.$$

2. Consider a smooth vector field defined in a connected domain, $D \subset \mathbb{R}^3$. Define what is meant by a path-independent vector field. Prove that any conservative vector field is path-independent.

   Given the vector field, $\mathbf{F} = (3x^2 + 6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^2\mathbf{k}$, evaluate $\int_C \mathbf{A} \cdot \mathbf{dl}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve C given by

   (a) $C = \{(x, y, z) \mid x = t, y = t^2, z = t^3\}$

   (b) the straight line from $(0, 0, 0)$ to $(1, 1, 1)$.

3. The Laplace transform of a function $f(t)$ is defined, for $t \geq 0$, as

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}dt,$$

   for $s \in \mathbb{C}$.

   Show that

$$\text{(i)}\ \mathcal{L}(\mathrm{e}^{\omega \mathrm{t}}) = \frac{1}{\mathrm{s} - \omega}, \text{ for } \mathrm{s} > \omega \text{ and (ii) } \mathcal{L}(\cos(\omega \mathrm{t})) = \frac{\mathrm{s}}{\mathrm{s}^2 + \omega^2}, \text{ for } \mathrm{Re(s)} > 0.$$

   The Laplace transform satisfies the derivative equation

$$\mathcal{L}(f') = sF(s) - f(0),$$

   for $f(t)$ a continuous function.

   Derive this expression, using integration by parts and assuming $\lim_{t\to\infty} f(t)e^{-st} = 0$.

   Use this property of the Laplace transforms to solve the differential equation

$$f' - f = 0,$$

   with inital condition $f(0) = 1$.

4. Consider the periodic function,

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ \pi, & \pi < x < 2\pi \end{cases}$$

with $f(x + 2\pi) = f(x)$.

Sketch the function $f(x)$ on the interval $-2\pi < x < 2\pi$.

Determine the Fourier series of $f(x)$.

5. Let S be the surface obtained by rotating the curve

$$\begin{aligned} r &= \cos(u) \\ z &= \sin(2u) \end{aligned}$$

for $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, around the z-axis.

Use the divergence theorem to find the volume of the region inside S.

6. Determine the Fourier integral representation of

$$f(x) = \begin{cases} e^{at} & t \leq 0 \\ e^{-at} & t \geq 0 \end{cases}$$

for $a > 0$.

Show that the fourier transform of the cosine function can be written as the sum of two delta functions.

7. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3}$$

has zero divergence, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Prove the identities

(a) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(b) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \triangle\mathbf{F}$.

8. Obtain the general solution of the ODE

$$y''(t) + 3y'(t) + 2y(t) = f(t)$$

for $f(t)$ a periodic function given by

$$f(t) = \begin{cases} 0 & -\pi < t < -a \\ 1 & -a < t < a \\ 0 & a < t < \pi \end{cases}$$

with $a \in (0, \pi)$ a constant and $f(t) = f(t + 2\pi)$.

1 Some useful formulae

1. A function with period l has a Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{l}\right),$$

where

$$\begin{aligned} a_0 &= \frac{2}{l}\int_{-l/2}^{l/2} dx\, f(x), \\ a_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx\, f(x)\cos\left(\frac{2\pi n x}{l}\right), \\ b_n &= \frac{2}{l}\int_{-l/2}^{l/2} dx\, f(x)\sin\left(\frac{2\pi n x}{l}\right). \end{aligned}$$

2. A function with period l has a Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2i\pi n x}{l}\right),$$

where

$$c_n = \frac{1}{l}\int_{-l/2}^{l/2} dx\, f(x)\exp\left(\frac{-2i\pi n x}{l}\right).$$

3. The Fourier integral representation (or Fourier transform) is

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} e^{ikx}, \\ \widetilde{f(k)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}. \end{aligned}$$