UNIVERSITY OF DUBLIN

XMA1M011

TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Natural Sciences **Trinity Term 2012**
JF Earth Sciences
JF Human Health and Disease

MA1M01 — Mathematical methods

??? — ??? (3 hours)

Dr. S. Ryan, Dr. E. Vergara-Diaz

Credit will be given for the best THREE questions answered IN EACH SECTION.
Use a different answer book for each section.

Formulae and log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination.
Please indicate the make and model of your calculator on each answer book used.

Section A

Credit will be given for the best 3 questions answered in this section.

1. Consider the system of linear equations

$$\begin{aligned} 5x - y &= 14 \\ 2x + 3y &= 26 \end{aligned}$$

   (a) [4 marks] Write the corresponding matrix equation.

   (b) [6 marks] By calculating the determinant of the matrix A in your matrix equation, determine that it is invertible.

   (c) [10 marks] Use Gauss-Jordan elimination to find the solution of this system of equations.

(a)

$$\begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 26 \end{pmatrix}.$$

(b) $\det(A) = 5(3) - (-1)(2) = 15 + 2 = 17$. Since $17 \neq 0$ the matrix **is** invertible.

(c)

$$\left(\begin{array}{cc|c} 5 & -1 & 14 \\ 2 & 3 & 26 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & -7 & -38 \\ 2 & 3 & 26 \end{array}\right)$$

$$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -7 & -38 \\ 0 & 17 & 102 \end{array}\right)$$

$$\xrightarrow{R_2/6} \left(\begin{array}{cc|c} 1 & -7 & -38 \\ 0 & 1 & 6 \end{array}\right)$$

$$\xrightarrow{R_1 + 7R_2} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right).$$

2. Consider an $n \times n$ matrix A.

   (a) [5 marks] Write down its characteristic equation and identify each term in the equation.

   (b) [15 marks] Given a matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

   Determine the eigenvalues of A and A^T and verify that they are equal.

(a) $\det(A - \lambda \mathbb{I}) = 0$. A is the matrix of coefficients, λ the eigenvalues of A and $\mathbb{I}$ the identity matrix.

(b) for A, determine λ from

$$det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = 0$$

$\Rightarrow (1-\lambda)(5-\lambda)-2(2) = 0$ ie $\lambda^2-6\lambda+5 = 0$ and factoring gives the eigenvalues $\lambda_1 = 5,\ \lambda_2 = 1$.
Now the transpose of A is

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

so the matrix is symmetric and the eigenvalues are the same.

3. (a) [6 marks] Define the addition and multiplication rules for probabilities.
   (b) [5 marks] Given two events A and B what is the conditional probability $P(B|A)$?
   (c) [9 marks] Consider a five-day school week. The probability that it is Friday and that a student is absent is 0.03. What is the probability that a student is absent given that today is Friday?

(a) **addition:** If A and B are disjoint events then $P(A or B) = P(A) + P(B)$
multiplication: Let A and B be two events then $P(A and B) = P(A)P(B|A)$.

(b) Conditional probability $P(B|A)$ is the probability event B happens given A has occurred.

(c) Considering a 5-day week, the probability today is Friday is $1/5 = 0.2$. The probability that it is Friday and a student is absent is 0.03. Then $P(Friday and Absent) = P(Friday)P(Absent|Friday)$ so

$$P(Absent|Friday) = P(Friday and Absent)/P(Friday) = 0.03/0.2 = 0.72 = 72\%.$$

4. Consider a continuous random variable, X, on an interval $[a, b]$.

   (a) [4 marks] State the three conditions a function f must satisfy to be a density function on this interval.

   (b) [8 marks] Verify that the following are probability density functions

   (i). $f(x) = 4x^3$ on $[0, 1]$.

   (ii). $f(x) = \frac{3}{26}x^2$ on $[1, 3]$.

   (c) [8 marks]

   (i). For $f(x) = 4x^3$ on $[0, 1]$, determine the probability $P(1/4 \leq X \leq 3/4)$.

   (ii). For $f(x) = \frac{3}{26}x^2$ on $[1, 3]$, determine the probability $P(1/4 \leq X \leq 3/4)$.

(a) Conditions are:

(i). f is nonnegative over $[a, b]$; that is $f(x) \geq 0 \forall x \in [a, b]$

(ii). $\int_a^b dx f(x) = 1$

(iii). $P(c \leq X \leq d) = \int_c^d dx f(x)$ for any subinterval $[c, d]$ of $[a, b]$

(b) (i). $f(x) = 4x^3$ on $[0, 1]$. This is always positive and $\int_0^1 dx 4x^3 = 4/4x^4|_0^1 = 1$.

(ii). $f(x) = 3/26x^2$ on $[1, 3]$. Always positive and $\int_1^3 3/26x^2 = 1/26x^3|_1^3 = 1/26(27 - 1) = 1$.

(c) (i). $P(1/4 \leq X \leq 3/4) = \int_{1/4}^{3/4} dx 4x^3 = x^4|_{1/4}^{3/4} = (3/4)^4 - (1/4)^4 = (81 - 1)/256 = 20/64 = 5/16 = 0.3125 \sim 32\%$.

(ii). $P(1/4 \leq X \leq 3/4) = \int_{1/4}^{3/4} dx 3/26x^2 = \frac{1}{26}x^3|_{1/4}^{3/4} = \frac{1}{26}\left[\left(\frac{3}{4}\right)^3 - \left(\frac{1}{4}\right)^3\right] = 1/64$.

Section B

Credit will be given for the best 3 questions answered in this section.

5. Let $f(x) = 3x - 5, x \in \mathbb{R}$ $g(x) = \frac{4}{6-x}, x \in \mathbb{R}, x \neq 6$, $h(x) = x^2 + 4x - 1$, $x \in \mathbb{R}$.

   (a) [5 marks] Find the value of $f(3), g(4), h(-1)$,

   (b) [6 marks] Compute $g(f(x))$, $h(f(x))$.

   (c) [9 marks] Sketch $f(x) = 3x - 5$, $h(x) = x^2 + 4x - 1$. State their domain and range.

6. Solve the following equations

   (a) [5 marks] $\ln(2x-5) - \ln x = \frac{1}{4}$

   (b) [5 marks] $e^{5t} = 4e^{2t+1}$.

   (c) [5 marks] $2\csc^2 x - \cot x - 12 = 0$

   (d) [5 marks] $\sin(x + \frac{4\pi}{3}) = \frac{\sqrt{2}}{2}, \qquad 0 \leq x + \frac{4\pi}{3} \leq 2\pi$

7. (a) Differentiate each of the following:

   (a) [2.5 marks] $y = \frac{e^x}{x^3 - 3}$

   (b) [2.5 marks] $y = \cos(5x+4)\ln(3x+1)$

   (b) [7 marks] Find the equations for the normal and the tangent to the curve $y = 3x^2 - e^{-2x}$ at the point where $x = 2$.

   (c) [8 marks] Find the coordinates of the stationary points of the curve $y = \frac{x^3}{3} - 4x$.

8. (a) Compute the following integrals

   (a) [5 marks] $\int \cos 4x \; \mathrm{d}x$

   (b) [5 marks] $\int xe^x \; \mathrm{d}x$

   (b) [10 marks] Find the area below $2x\cos(x^2)$, between $x = 0$ and $x = 1$.