UNIVERSITY OF DUBLIN

XMA1M011

TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Natural Sciences **Trinity Term 2011**
JF Earth Sciences
JF Human Health and Disease

MA1M01 — Mathematical methods

??? — ??? (3 hours)

Dr. S. Ryan, Dr. R. Levene

Credit will be given for the best THREE questions answered IN EACH SECTION.
Use a different answer book for each section.

Formulae and log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination.
Please indicate the make and model of your calculator on each answer book used.

Section A

Credit will be given for the best 3 questions answered in this section.

1. [10 marks] Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix}.$$

determine eigenvalues from $det(A - \lambda\mathbb{I}) = 0$

$$\left| \begin{pmatrix} 6-\lambda & 16 \\ -1 & -4-\lambda \end{pmatrix} \right| = 0$$

giving $(6-\lambda)(-4-\lambda)+16 = (\lambda^2 - 2\lambda - 8) = 0$. This factorises as $(\lambda-4)(\lambda+2) = 0$ so the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -2$.

Using $Av = \lambda v$ to find the eigenvalue v for $\lambda = 4$:

$$\begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

so, $6v_1+16v_2 = 4v_1$ and $-v_1-4v_2 = 4v_2$ and these are equivalent, giving $v_1 = -8v_2$. So, an eigenvalue is $\begin{pmatrix} -8 \\ 1 \end{pmatrix}$.

Using $Av = \lambda v$ to find the eigenvalue v for $\lambda = -2$:

$$\begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

so, $6v_1 + 16v_2 = -2v_1$ and $-v_1 - 4v_2 = -2v_2$ and these are equivalent, giving $v_1 = -2v_2$. So, an eigenvalue is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

[10 marks] Using $A^k = PD^kP^{-1}$, where P is the matrix whose columns are the eigenvectors and D is the diagonal matrix of eigenvalues, find A^4. so,

$$P = \begin{pmatrix} -8 & -2 \\ 1 & 1 \end{pmatrix} \quad so \;\; P^{-1} = \frac{1}{-6}\begin{pmatrix} 1 & 2 \\ -1 & -8 \end{pmatrix}.$$

and

$$D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then

$$\begin{aligned}
A^4 &= \begin{pmatrix} -8 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}^4 \frac{1}{-6}\begin{pmatrix} 1 & 2 \\ -1 & -8 \end{pmatrix} \\
&= \begin{pmatrix} -8 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 256 & 0 \\ 0 & 16 \end{pmatrix}\frac{1}{-6}\begin{pmatrix} 1 & 2 \\ -1 & -8 \end{pmatrix} \\
&= \begin{pmatrix} -2048 & -32 \\ 256 & 16 \end{pmatrix}\frac{1}{-6}\begin{pmatrix} 1 & 2 \\ -1 & -8 \end{pmatrix} \\
&= \frac{1}{-6}\begin{pmatrix} -2016 & -3840 \\ 240 & 384 \end{pmatrix} \\
&= \begin{pmatrix} 336 & 640 \\ -40 & -64 \end{pmatrix}.
\end{aligned}$$

agrees with Cramer's rule.

2. [6 marks] Demonstrate that the matrix

$$A = \begin{pmatrix} -12 & -7 \\ -7 & 10 \end{pmatrix}$$

is symmetric (ie $A^T = A$) and invertible.

[14 marks] Use Gauss-Jordan elimination to determine A^{-1}, the inverse of A.

[2 marks] Show that A^{-1} is also symmetric.

$$A^T = \begin{pmatrix} -12 & -7 \\ -7 & 10 \end{pmatrix}.$$

so A symmetric. $\det(A) = -120 + 49 \neq 0$ so A invertible.

$$\left(\begin{array}{cc|cc} -12 & -7 & 1 & 0 \\ -7 & 10 & 0 & 1 \end{array}\right) \overset{R_1/-12}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & 7/12 & -\frac{1}{12} & 0 \\ -7 & 10 & 0 & 1 \end{array}\right)$$

$$\overset{R_2+7R_1}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & \frac{7}{12} & -\frac{1}{12} & 0 \\ 0 & \frac{169}{12} & -\frac{7}{12} & 1 \end{array}\right)$$

$$\overset{12R_2/169}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & \frac{7}{12} & -\frac{1}{12} & 0 \\ 0 & 1 & -\frac{7}{169} & \frac{12}{169} \end{array}\right)$$

$$\overset{R_1-7/12R_2}{\longrightarrow} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{10}{169} & -\frac{7}{169} \\ 0 & 1 & -\frac{7}{169} & \frac{12}{169} \end{array}\right)$$

3. [5 marks] Consider the numbers below. In each case state if the number can or cannot be a probability.

a) -0.00001; b) 0.5; c) 1.001; d) 0; e) 1; a) no; b) yes; c) no; d) yes; e) yes.

(a) [5 marks] A die is rolled, find the probability that the number obtained is greater than 4.

(b) [5 marks] Two dice are rolled, find the probability that the sum is equal to 5.

(c) [5 marks] A die is rolled and a coin is tossed, find the probability that the die shows an odd number and the coin shows a head.

i) from 6 possible outcomes, 2 are greater than 4 so P(number¿4)=2/6=1/3.
ii) total number of possible outcomes = 6x6=36, P(sum =5)=4/36=1/9.
iii) P(A and B) = P(A)P(B) = (3/6)(1/2) = 1/4.

4. [6 marks] Consider a continuous random variable, X, on an interval $[a, b]$. State the three conditions a function f must satisfy to be a density function on this interval.

A 3cc solution contains exactly one bacterium. The solution is slowly transferred into a flask using a pipette. The random variable X is the amount of solution already transferred to the flask when the bacterium enters (so X is a continuous random variable over the interval $[0, 3]$). Suppose the density function is

$$f(x) = \frac{1}{9}x^2.$$

(a) [5 marks] Compute the probability that the bacterium enters the flask with the first cc of transferred solution.

(b) [5 marks] Compute the probability that the bacterium enters the flask with the last cc of solution.

(c) [4 marks] By sketching $f(x)$ explain the difference in magnitude of the probabilities you have computed.

Conditions of f: i) f is non-negative on the interval. ii) $\int_a^b f(x)dx = 1$ and iii) $P(c \leq X \leq d) = \int_c^d f(x)dx$ for any subinterval $[c, d]$ of $[a, b]$.

Probability it enters in first cc:
Want

$$P(0 \leq X \leq 1) = \int_0^1 f(x)dx = \int_0^1 \frac{1}{9}x^2 dx = \frac{1}{9}\frac{1}{3}x^3\Big|_0^1 = \frac{1}{27}(1^3 - 0^3) = \frac{1}{27}.$$

ie. probability is $1/27 \approx 0.037$.

Probability it enters in last cc:
Want

$$P(2 \leq X \leq 3) = \int_2^3 \frac{1}{9}x^2 dx = \frac{1}{27}x^3\Big|_2^3 = \frac{1}{27}(3^3 - 2^3) = \frac{19}{27}.$$

ie probability is $19/27 \approx 0.7037$.