UNIVERSITY OF DUBLIN XMA1M01

TRINITY COLLEGE

FACULTY OF ENGINEERING, MATHEMATICS
AND SCIENCE

SCHOOL OF MATHEMATICS

JF Natural Sciences **Trinity Term 2010**
JF Human Health and Disease

MODULE 1M01

Wednesday, May 12 Sports Centre 09:30–12:30

Dr. S. Ryan, Dr. R.H. Levene

Credit will be given for the best THREE questions attempted IN EACH SECTION.
Use a different answer book for each section.

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination.
Please indicate the make and model of your calculator on each answer book used.

SECTION A

1. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}.$$

(a) [14 marks] Use Gauss-Jordan elimination to find the inverse of the matrix A.

(b) [3 marks] Verify that your result is correct.

(c) [3 marks] Verify that you obtain the same result by applying Cramer's rule.

Write the augmented matrix and use row operations to determine A^{-1}

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 3/2R_1} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right)$$

$$\xrightarrow{2R_2} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 - 3R_2} \left(\begin{array}{cc|cc} 2 & 0 & 10 & -6 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\xrightarrow{R_1/2} \left(\begin{array}{cc|cc} 1 & 0 & 5 & -3 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

So,

$$A^{-1} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}.$$

To check this is correct, check $AA^{-1} = A^{-1}A = I$, ie

$$AA^{-1} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2(5) + 3(-3) & 2(-3) + 3(2) \\ 3(5) + 5(-3) & 3(-3) + 5(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

and similarly for $A^{-1}A$.

By Cramer's rule, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } A^{-1} = \frac{1}{detA} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Checking, $det(A) = 2(5) - 3(3) = 1$ so

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}.$$

as before.

2. Consider the following statement:

Theorem: Suppose the largest eigenvalue of a Leslie matrix is r. Then the relative growth rate of the population described approaches r as the number of years approaches infinity ie the long-term growth rate is the largest eigenvalue of the system.

Now, given the Leslie matrix for a particular bird population in central Missouri is

$$G = \begin{pmatrix} 0.728 & 1.302 \\ 0.52 & 0.62 \end{pmatrix}.$$

(a) [6 marks] What is the defining equation of an eigenvalue λ of an $n \times n$ matrix A?

(b) [14 marks] Find the long-term (percentage) growth rate from the Leslie matrix G given here.

The characteristic equation is $det(A - \lambda I) = 0$ and $Av = \lambda v$.

For % growth rate, first find eigenvalues from $det(A - \lambda I) = 0$.

$$\begin{aligned} 0 &= det\begin{pmatrix} 0.728 - \lambda & 1.302 \\ 0.52 & 0.62 - \lambda \end{pmatrix} \\ &= (0.728 - \lambda)(0.62 - \lambda) - (0.52)(1.302) \\ &= 0.45136 - 1.348\lambda + \lambda^2 - 0.67704 \\ &= \lambda^2 - 1.348\lambda - 0.22568 \end{aligned}$$

Finding roots gives

$$\begin{aligned} \lambda &= \frac{-(-1.348) \pm \sqrt{(-1.348)^2 - 4(1)(-0.22568)}}{2(1)} \\ &= \frac{1.348 \pm \sqrt{2.719824}}{2} \\ &= (1.348 \pm 1.6492)/2 \end{aligned}$$

So, the eigenvalues are $\lambda_1 = 1.4985$ and $\lambda_2 = -0.1506$. The largest eigenvalue is $\lambda_1 = 1.4985$ giving a % growth rate of 49.85% ie ≈50%.

3. Consider the following experiment. A box contains five tickets labelled

1	0	0	0	1

Six tickets are drawn randomly from this box with replacement. That means that a ticket is drawn, observed and then replaced before the next ticket is drawn.

(a) [5 marks] Define a binomial experiment.

(b) [5 marks] Explain why the experiment described is binomial.

(c) [5 marks] Find the probability that the sum of the six tickets drawn is 3.

(d) [5 marks] What is the standard deviation for this experiment?

A binomial experiment is one for which

- there is a fixed number, n of trials.
- each trial can be characterised a success or failure.
- the probability for success in each trial is constant/
- the central problem is to count the number of successes that occur.

The experiment above is binomial since: there is a fixed number of trials ie six tickets are drawn; each trial is either a success (a 1 is drawn) or a failure (a 0 is drawn); the probability for success is constant, ie for each trial P(draw 1) =2/5; the sum of tickets is 3 only if exactly 3 tickets labelled 1 are drawn.

To find the probability: let X be the number of 1s drawn. X has a binomial (6,2/5) distribution ie $n = 6, p = 0.4$ and $q = 1 - p = 0.6$. So,

$$\begin{aligned} P(X = k) &= \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k} \\ P(X = 3) &= \begin{pmatrix} 6 \\ 3 \end{pmatrix} (0.4)^3 (0.6)^{6-3} \\ &= \frac{6!}{3!3!}(0.064)(0.216) \\ &= (20)(0.064)(0.216) \\ &= 0.27648 \end{aligned}$$

so, an $\approx$ 28% probability that the sum of six tickets drawn is 3.

4. Consider a large class of students. Each student in the class is given an identical box containing three tickets, marked $\diamondsuit$, $\heartsuit$ and $\square$. Students are asked to draw two tickets from the box without replacing the first ticket back into the box. A prize is given to those who draw a $\diamondsuit$ first and a $\square$ second.

   (a) [3 marks] State the principle of equally likely outcomes.

   (b) [5 marks] State the multiplication rule for the probability that two events, A and B, both occur. Explain what the probability $P(B|A)$ means.

   (c) [4 marks] Considering the experiment described above, what proportion of students draw the $\diamondsuit$ ticket first?

   (d) [4 marks] Of the students that drew the $\diamondsuit$ first, what proportion draw the $\square$ second?

   (e) [4 marks] What proportion of the students in the class will win the prize?

Principle of equally likely outcomes: *from notes*

Multiplication rule: P(A and B) = P(A)P(B—A)

$P(B|A)$ is the conditional probability that B occurs given A has occured.

P($\diamondsuit$) $= 1/3$: a third of the class draw a $\diamondsuit$.

P(these students draw a $\square$) $= 1/2$: half the students that drew a $\diamondsuit$ will draw a $\square$.

P($\diamondsuit$ first followed by $\square$) $= (1/3)(1/2) = 1/6$: a sixth of the class win the prize.

SECTION B

5. (a) [6 marks] Find an equation for the straight line which is parallel to $y = 3x - 4$ and passes through the point $(1, 2)$.

   (b) [7 marks] Simplify the expression $\sqrt[4]{16x^3 e^{\ln(x)}}$.

   (c) [7 marks] What is the average rate of change of the function $f(x) = x^2 - 3x$ as x varies from -1 to 2?

6. (a) Find the derivatives of the following expressions:

   (i). [4 marks] $\dfrac{\sin(x)}{1+x}$

   (ii). [4 marks] $5 - 6e^{1+x^2} + x\ln(x)$

   (iii). [4 marks] $(x^2 + \sin(2\pi x))^{0.3}$

   (b) [8 marks] Evaluate the indefinite integral $\displaystyle\int e^x - 4x\sin(x^2)\,dx$.

7. Let $f(x) = 12x - x^3 + 10$.

   (a) [10 marks] Compute the coordinates of the absolute maximum and the absolute minimum of $f(x)$ over the interval $[0, 3]$.

   (b) [10 marks] What is the average value of $f(x)$ over the interval $[0, 3]$?

8. The number of bacteria $P(t)$ in a Petri dish grows according to

$$P(t) = (3 \times 10^5)e^{0.04t}$$

where t is the time, in hours, since the bacteria were introduced to the Petri dish.

   (a) [2 marks] How many bacteria were introduced to the Petri dish?

   (b) [4 marks] What is the generation time for this population?

   (c) [7 marks] Compute the instantaneous rate of change of $P(t)$ 24 hours after the bacteria were introduced to the Petri dish.

   (d) [7 marks] When does the population reach 2×10^6 bacteria?