UNIVERSITY OF DUBLIN

XMA

TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Natural Sciences **Trinity Term 2009**

Module 1M01

9.30 — 12.30

Dr. S. Ryan, Dr. R.H. Levene

Credit will be given for the best THREE questions attempted IN EACH SECTION.
Use a different answer book for each section.

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination.
Please indicate the make and model of your calculator on each answer book used.

SECTION A

1. Consider the system of linear equations

$$\begin{aligned} 2x + y &= 3, \\ x + 2y &= 4. \end{aligned}$$

Express this linear system as a matrix equation of the form $Ax = b$.

Calculate the determinant of the matrix A for this linear system.

Use Gauss-Jordan elimination to solve for the parameters x and y.

The matrix equation is

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

$$\begin{aligned} \texttt{det} A &= \texttt{det} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= (2)(2) - (1)(1) \\ &= 3. \end{aligned}$$

Here is a possible GJ solution (not unique).

$$\left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right) \overset{R2-R1/2}{=} \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 3/2 & 5/2 \end{array}\right)$$
$$\overset{R2*2/3}{=} \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & 5/3 \end{array}\right)$$
$$\overset{R1-R2}{=} \left(\begin{array}{cc|c} 2 & 0 & 4/3 \\ 0 & 1 & 5/3 \end{array}\right)$$
$$\overset{R1/2}{=} \left(\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & 5/3 \end{array}\right)$$

So, $x = 2/3$ and $y = 5/2$ and you should check that these solutions are correct by substituting in the original equations.

2. Write down the characteristic equation of a matrix A.

The Leslie matrix, A, for a bird population of hatchlings and adults is given by

$$A = \begin{pmatrix} 0.5 & 2 \\ 0.5 & 0.5 \end{pmatrix}.$$

Find the eigenvalues of this matrix.

Recall that the long-term growth rate of a population is given by the largest eigenvalue of the corresponding Leslie matrix and the stable age population is given its corresponding eigenvector.

Determine the stable-age population for the Leslie matrix given above.

Characteristic equation: $\texttt{det}(A - \lambda I) = 0$.

Find the eigenvalues using the characteristic equation.

$$\begin{aligned} \texttt{det}(A - \lambda I) = 0 &= \texttt{det}\begin{pmatrix} 0.5-\lambda & 2 \\ 0.5 & 0.5-\lambda \end{pmatrix} \\ &= (0.5-\lambda)(0.5-\lambda) - 1 \\ &= \lambda^2 - \lambda - \frac{3}{4} \\ &= (\lambda + \frac{1}{2})(\lambda - \frac{3}{2}) \end{aligned}$$

so the eigenvalues are $\lambda = -1/2$ and $\lambda = 3/2$.

The stable-age population is the eigenvector corresponding to $\lambda = 3/2$.

We have $Av = \lambda v$ and need the v that solves this for $\lambda = 3/2$.

So,

$$\begin{pmatrix} 0.5 & 2 \\ 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Doing the matrix multiplication gives 2 equations in 2 unknowns ie.

$$\begin{aligned} 0.5v_1 + 2v_2 &= 1.5v_1 \\ 0.5v_1 + 0.5v_2 &= 1.5v_2 \end{aligned}$$

Simplifying gives a single equation relating v_1 and v_2 namely $v_1 = 2v_2$ and any numbers satisfying this form the correct eiqenvector. So, eg. if $v_2 = 1$ then $v_1 = 2$ and the eigenvector is $(2, 1)$. The stable-age populations is hatchlings to adults in a ratio 2:1.

3. Define a binomial experiment.

   Write down the formula for computing the probability of success in a binomial experiment.

   A survey finds that 30% of teenage consumers receive their spending money from part-time jobs. If five teenagers are selected at random, find the probability that at least three of them will have part-time jobs.

   A binomial experiment is one in which

   - a trial is repeated a fixed number, n, times.
   - each trial is independent
   - each trial results in only two outcomes: success and failure.
   - the probability of success, P(success)= p, is constant.
   - the goal of the experiment is to count the number of successes in n trials.

   Formula for succes in a binomial expt is

$$P(X = k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k}.$$

   To find probability at least 3 have a job need $P(X = 3) + P(X = 4) + P(X = 5)$, using

$$P(X = k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k}.$$

   Here, $= 0.3$ (ie 30%), so $q = 1 - p = 0.7$ and $n = 5$. So,

$$\begin{aligned} P(X = 3) &= \begin{pmatrix} 5 \\ 3 \end{pmatrix} (0.3)^3 (0.7)^2 = 0.1323 \\ P(X = 4) &= \begin{pmatrix} 5 \\ 4 \end{pmatrix} (0.3)^4 (0.7)^1 = 0.02835 \\ P(X = 5) &= \begin{pmatrix} 5 \\ 5 \end{pmatrix} (0.3)^5 (0.7)^0 = 0.00243 \end{aligned}$$

   and summing these gives 0.16308, ie a 16% probability 3 or more have jobs.

4. By giving the definitions of independent and disjoint events, explain how they are different.

   A fair die is rolled four times.

   (a) what is the probability that the rolls are all sixes?

   (b) what is the probability that the rolls are all the same?

   (c) what is the probability that the rolls are all different?

(d) what is the probability that a face (one particular side) appears more that once?

Independent: 2 events A and B are independent if the occurence of one gives no information about whether or not the other will occur.

Disjoint: 2 events are disjoint if it is impossible for them to occur together.

Appeared in PS10.

- P(all rolls are sixes) $= \frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6} = \frac{1}{1296}$.
- there are six ways (ie 6 different numbers) that can all be the same so P(all rolls the same) $= \frac{6}{1296}$.
- P(all rolls are different) $= 1\frac{5}{6}\frac{4}{6}\frac{3}{6} = \frac{5}{18}$.
- P(face appears more than once) = P(not all rolls different) = 1- P(all rolls different) $= 1 - \frac{5}{18} = \frac{13}{18}$.

SECTION B

5. (a) Convert the angle $25°$ into radians.
   (b) Find the equation of the straight line which joins the points $(1, 1)$ and $(2, -3)$.
   (c) Find the slope of the tangent line to $y = \dfrac{\ln x}{1 + x}$ at $x = 1$.

6. (a) Compute the definite integral

$$\int_0^1 x(x^2 + 1)^{15}\, dx.$$

   (b) Find the minimum value of $(x + 1)^2 + y^2$ where x and y are real numbers with $x + y = 1$. Justify your answer.

7. Let $f(x) = 1 + 2\cos(\pi x)$. Compute the following:
   (a) The average rate of change of $f(x)$ over the interval $[0, 3]$.
   (b) The instantaneous rate of change of $f(x)$ at $x = 3$.
   (c) The average value of $f(x)$ over the interval $[0, 3]$.

8. A man's heart rate $h(t)$, in heartbeats per second, is measured by a scientist over the course of an experiment. Here t is the time, in seconds, since the start of the experiment. When $t = 100$, a dose of adrenaline is administered.

   The scientist finds that $h(t)$ is given by

$$h(t) = \begin{cases} 1.1 & \text{for } 0 \leq t \leq 100 \\ 1.1 + e^{1-0.01t} & \text{for } t > 100. \end{cases}$$

   (a) Is the function $h(t)$ continuous at $t = 100$? Justify your answer.
   (b) Find the time t for which $h(t) = 1.6$.
   (c) Explain briefly why the number of times the man's heart beats between $t = 0$ and $t = 200$ is given by

$$110 + \int_{100}^{200} 1 + e^{1-0.01t}\, dt.$$

   You should *not* find the value of this expression.