MAU11002: Mathematics Tutorial Sheet [1]

1. Consider the 2×2 Leslie matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0.25 & 0.5 \end{pmatrix}.$$

   (a) Given a population vector $P = \begin{pmatrix} 100 \\ 160 \end{pmatrix}$ in year N, use matrix-vector multiplication to determine the population in year $N+1$.

   (b) Determine the eigenvalues of A.

   (c) Recall that the long-term growth rate of a population is given by the largest eigenvalue of the corresponding Leslie matrix and the stable age population is given by its corresponding eigenvector.

   Determine the stable-age population for the Leslie matrix, A, given above.

The population in $N+1$ is given by AP and is $\begin{pmatrix} 320 \\ 105 \end{pmatrix}$.

The eigenvalues are $\lambda_1 = -1/2, \lambda_2 = 1$ and the stable age is the eigenvector for $\lambda = 1$ ie $v = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix}$.

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2. Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix}.$$

Starting from $AP = PD$ where P is the matrix whose columns are the eigenvectors and D is the diagonal matrix of eigenvalues, show that $A^k = PD^kP^{-1}$.

Find A^4.

Determine eigenvalues from $det(A - \lambda I) = 0$

$$\left| \begin{pmatrix} 6-\lambda & 16 \\ -1 & -4-\lambda \end{pmatrix} \right| = 0$$

giving $(6-\lambda)(-4-\lambda) + 16 = (\lambda^2 - 2\lambda - 8) = 0$. This factorises as $(\lambda - 4)(\lambda + 2) = 0$ so the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -2$.

Using $Av = \lambda v$ to find the eigenvalue v for $\lambda = 4$:

$$\begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

so, $6v_1 + 16v_2 = 4v_1$ and $-v_1 - 4v_2 = 4v_2$ and these are equivalent, giving $v_1 = -8v_2$. So, an eigenvalue is $\begin{pmatrix} -8 \\ 1 \end{pmatrix}$.

Using $Av = \lambda v$ to find the eigenvalue v for $\lambda = -2$:

$$\begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

so, $6v_1 + 16v_2 = -2v_1$ and $-v_1 - 4v_2 = -2v_2$ and these are equivalent, giving $v_1 = -2v_2$. So, an eigenvalue is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

For A^4:

$$P = \begin{pmatrix} -8 & -2 \\ 1 & 1 \end{pmatrix} \;\; so \;\; P^{-1} = \frac{1}{-6} \begin{pmatrix} 1 & 2 \\ -1 & -8 \end{pmatrix}.$$

and

$$D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then

$$
\begin{aligned}
A^4 &= \begin{pmatrix} -8 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}^4 \frac{1}{-6} \begin{pmatrix} 1 & 2 \\ -1 & -8 \end{pmatrix} \\
&= \begin{pmatrix} -8 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 256 & 0 \\ 0 & 16 \end{pmatrix} \frac{1}{-6} \begin{pmatrix} 1 & 2 \\ -1 & -8 \end{pmatrix} \\
&= \begin{pmatrix} -2048 & -32 \\ 256 & 16 \end{pmatrix} \frac{1}{-6} \begin{pmatrix} 1 & 2 \\ -1 & -8 \end{pmatrix} \\
&= \frac{1}{-6} \begin{pmatrix} -2016 & -3840 \\ 240 & 384 \end{pmatrix} \\
&= \begin{pmatrix} 336 & 640 \\ -40 & -64 \end{pmatrix}.
\end{aligned}
$$

3. Use the Cayley Hamilton theorem to determine A^5 given,

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \tag{1}$$

Write down an expression for A^4.

Cayley Hamilton says a matrix satisfies its own characteristic equation. In this case: $det(A - \lambda I) = 0$ yields $\lambda^2 - 7\lambda + 10 = 0$ so we have that $A^2 - 7A + 10I = 0$.

$$\begin{aligned}
A^2 &= 7A - 10I \\
A^3 &= 7A^2 - 10A = 7(7A - 10I) - 10A = 39A - 70I \\
A^4 &= A(39A - 70I) = 39A^2 - 70A = 39(7A - 10I) - 70A = 203A - 390I \\
A^5 &= A(203A - 390I) = 203A^2 - 390A = 203(7A - 10I) - 390A = 1031A - 2030I
\end{aligned}$$

so that

$$\begin{aligned}
A^5 &= 1031 \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} - 2030 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3093 & 1031 \\ 2062 & 4124 \end{pmatrix} - \begin{pmatrix} 2030 & 0 \\ 0 & 2030 \end{pmatrix} \\
&= \begin{pmatrix} 1063 & 1031 \\ 2062 & 2094 \end{pmatrix}.
\end{aligned}$$