Q1 from SUMMER 2015

a) $\det A = 4(-2) - (1)(1) = -8 - 1 = -9$

$-9 \neq 0 \Rightarrow$ mx A invertible.

b) EROs:

i) Swap a row with any other row

ii) multiply all entries in a row by any non-zero number.

iii) add a multiple of any row to another row.

c) $\begin{matrix} 4x + y = 1 \\ x - 2y = 3 \end{matrix}$ has mx. eq$^n$ $\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

GJE:

$$\left(\begin{array}{cc|c} 4 & 1 & 1 \\ 1 & -2 & 3 \end{array}\right) \xrightarrow{4R_2 - R_1} \left(\begin{array}{cc|c} 4 & 1 & 1 \\ 0 & -9 & 11 \end{array}\right) \xrightarrow{R_2/(-9)} \left(\begin{array}{cc|c} 4 & 1 & 1 \\ 0 & 1 & -11/9 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c} 4 & 0 & 20/9 \\ 0 & 1 & -11/9 \end{array}\right) \xrightarrow{R_1/4} \left(\begin{array}{cc|c} 1 & 0 & 5/9 \\ 0 & 1 & -11/9 \end{array}\right)$$

$\Rightarrow x = 5/9; \quad y = -11/9$

CHECK:

eq$^n$ 1. $4\left(\frac{5}{9}\right) - \frac{11}{9} = \frac{20}{9} - \frac{11}{9} = 1$ ✓

eq$^n$ 2. $\frac{5}{9} - 2\left(-\frac{11}{9}\right) = \frac{5}{9} + \frac{22}{9} = \frac{27}{9} = 3$ ✓

Q2 SUMMER 2018

a) $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ with $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$; $\cancel{x}\; b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

b) $\det A = 3(2) - 1(4) = 6 - 4 = 2$

$2 \neq 0 \Rightarrow A$ invertible

c) for $A^{-1}$:

$$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_1/3} \left(\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 4R_1} \left(\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & \frac{2}{3} & \frac{-4}{3} & 1 \end{array}\right)$$

$$\xrightarrow{\frac{3}{2}\cdot R_2} \left(\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & 1 & -2 & 3/2 \end{array}\right) \xrightarrow{R_1 - \frac{1}{3}R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -2 & 3/2 \end{array}\right)$$

$\Rightarrow A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & 3/2 \end{pmatrix}$

a quick check

$\begin{pmatrix} 1 & -1/2 \\ -2 & 3/2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

& $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1/2 \\ -2 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

Finally since

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b$$

$$\underbrace{\phantom{A^{-1}A}}_{I} x = A^{-1}b \Rightarrow x = A^{-1}b$$

$A^{-1}b = \begin{pmatrix} 1 & -1/2 \\ -2 & 3/2 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 7/2 \end{pmatrix}$ ie $x = -1/2$, $y = 7/2$

CHECK $3(-1/2) + 7/2 = 4/2 = 2$ ✓

$4(-1/2) + 2(7/2) = -2 + 7 = 5$ ✓

Q3 SUMMER 2014

a) $A^T = A$ if $A = \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix} = A$ ✓

b) <u>for $A^{-1}$</u>

$$\left(\begin{array}{cc|cc} 4 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right) \xrightarrow{4R_2 - R_1} \left(\begin{array}{cc|cc} 4 & 1 & 1 & 0 \\ 0 & -9 & -1 & 4 \end{array}\right) \xrightarrow{-\frac{1}{9}R_2} \left(\begin{array}{cc|cc} 4 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{9} & -\frac{4}{9} \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \left(\begin{array}{cc|cc} 4 & 0 & 8/9 & 4/9 \\ 0 & 1 & 1/9 & -4/9 \end{array}\right) \xrightarrow{\frac{1}{4}R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{9} & 1/9 \\ 0 & 1 & 1/9 & -4/9 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 2/9 & 1/9 \\ 1/9 & -4/9 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}$$

check e.g. $AA^{-1} = \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}\frac{1}{9} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^{-1}A = I$ also.

c) then

$$A^{-1}A^T = \underbrace{\frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}}_{A^{-1}}\underbrace{\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}}_{A^T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

expected since:

$A^{-1}A^T = A^{-1}A = I$ for $A^T = A$.