MAU11002: Mathematics Tutorial Sheet 2 [1]

1. **Warm-up matrix multiplication exercise**
   Determine the following matrix products: AB, BC given

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 7 \end{pmatrix}; \; B = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 5 & 5 \\ 6 & 4 & 3 & 2 \end{pmatrix}; \; C = \begin{pmatrix} 2 \\ 6 \\ 9 \\ 1 \end{pmatrix}.$$

   Write down in advance the size of the product matrices, AB and BC.

   A is a 2×3 matrix; B is 3×4 and C is 4×1 so AB is 2×4 and BC is 3×1.

   Doing the multiplication

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 5 & 5 \\ 6 & 4 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 7 & 9 \\ 55 & 34 & 40 & 37 \end{pmatrix}.$$

   and

$$BC = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 3 & 2 & 5 & 5 \\ 6 & 4 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} 13 \\ 62 \\ 65 \end{pmatrix}.$$

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2. **Leslie matrices**

   Given a Leslie matrix, G, and a population vector v written

$$G = \begin{pmatrix} 1.2 & 2.5 \\ 0.8 & 0.8 \end{pmatrix} \text{ and } v = \begin{pmatrix} 240 \\ 124 \end{pmatrix}, \tag{1}$$

   Determine the population in years 2 and 3.

   In year 2 the population is given by

$$\begin{aligned} Gv &= \begin{pmatrix} 1.2 & 2.5 \\ 0.8 & 0.8 \end{pmatrix} \begin{pmatrix} 240 \\ 124 \end{pmatrix} \\ &= \begin{pmatrix} 1.2(240) + 2.5(124) \\ 0.8(240) + 0.8(124) \end{pmatrix} = \begin{pmatrix} 598 \\ 291.2 \end{pmatrix} \end{aligned}$$

   and in year 3 by

$$\begin{aligned} GGv &= \begin{pmatrix} 1.2 & 2.5 \\ 0.8 & 0.8 \end{pmatrix} \begin{pmatrix} 598 \\ 291.2 \end{pmatrix} \\ &= \begin{pmatrix} 1445.6 \\ 711.36 \end{pmatrix} \end{aligned}$$

3. **Matrix properties**
The *identity matrix* is a square matrix with 1's on the diagonal and zero elsewhere, eg the 2×2 identity matrix is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that $IA = AI = A$ using

$$A = \begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix}.$$

$$\begin{aligned} IA &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1(6)+0(2) & 1(4)+0(5) \\ 0(6)+1(2) & 0(4)+1(5) \end{pmatrix} \\ &= \begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix} = A. \end{aligned}$$

and

$$\begin{aligned} AI &= \begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6(1)+4(0) & 6(0)+4(1) \\ 2(1)+5(0) & 2(0)+5(1) \end{pmatrix} \\ &= \begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix} = A. \end{aligned}$$

4. **Matrix properties again**
Consider the matrices

$$A = \begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix}; \; B = \begin{pmatrix} 3 & 5 \\ -4 & 1 \end{pmatrix}.$$

By writing down the transpose of A and B, state if either is symmetric.

Show by direct calculation that $(A^T)^T = A$ and $(AB)^T = B^T A^T$.

Swopping rows and columns in A and B

$$A^T = \begin{pmatrix} 6 & 2 \\ 4 & 5 \end{pmatrix}; \; B^T = \begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix}.$$

In both cases: $A \neq A^T$ and $B \neq B^T$ so neither is symmetric

Now

$$(A^T)^T = \begin{pmatrix} 6 & 2 \\ 4 & 5 \end{pmatrix}^T = \begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix} = A$$

as required.

$$AB = \begin{pmatrix} 6 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 34 \\ -14 & 15 \end{pmatrix}.$$

and so

$$(AB)^T = \begin{pmatrix} 4 & -14 \\ 34 & 15 \end{pmatrix}.$$

Also, using the transposes written down earlier

$$B^T A^T = \begin{pmatrix} 3 & -4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -14 \\ 34 & 15 \end{pmatrix}$$

Proving that for matrices A and B given here, $(AB)^T = B^T A^T$.