1MA01: Linear Algebra

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In this section on linear algebra we discuss properties of *matrices* and examples of their application eg to population growth.

Matrix

A compact wat of representing or storing information organised in a rectangular array of numbers. A matrix is usually (in these notes) denoted with a capital letter.
If a matrix A has n rows and m columns then its size is $n \times m$.

E.g.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \text{ isa } 2 \times 3 \text{ matrix}$$

and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \\ 2 & -3 \end{pmatrix} \text{ isa } 3 \times 2 \text{ matrix}$$

If $n = m$ ie number of rows = number of columns then it is called a *square matrix*
E.g.

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix} \text{ isa } 2 \times 2 \, \text{matrix}$$

To refer to a number in a matrix A (called an entry) use its row and column postition e.g. entry a_{23} is found at row 2 and column 3.
E.g.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \text{ then } a_{23} = 1$$

Note that entries can be integers ($\pm$), real numbers or complex numbers (or a mixture).

Matrix Addition and Subtraction

Define the sum/difference of two matrices *of the same size* by adding/subtracting corresponding entries.

definition

A, B two $n \times m$ matrices.
Then $C = A + B$ is also $n \times m$ and $c_{ij} = a_{ij} + b_{ij}$.
Also $D = A - B$ is also $n \times m$ and $d_{ij} = a_{ij} - b_{ij}$.

E.g. for the three matrices

$$A = \begin{pmatrix} 3 & 0 & -6 \\ -1 & 7 & -3 \end{pmatrix}; B = \begin{pmatrix} 4 & -2 & 0 \\ 5 & -2 & 10 \end{pmatrix}; C = \begin{pmatrix} 3 & 9 \\ 2 & 1 \end{pmatrix}$$

we compute:

$$
\begin{aligned}
A+B &= \begin{pmatrix} 3 & 0 & -6 \\ -1 & 7 & -3 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 0 \\ 5 & -2 & 10 \end{pmatrix} \\
&= \begin{pmatrix} 3+4 & 0-2 & -6+0 \\ -1+5 & 7-2 & -3+10 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -2 & 6 \\ 4 & 5 & 7 \end{pmatrix};
\end{aligned}
$$

$$
\begin{aligned}
A-B &= \begin{pmatrix} 3 & 0 & -6 \\ -1 & 7 & -3 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 0 \\ 5 & -2 & 10 \end{pmatrix} \\
&= \begin{pmatrix} 3-4 & 0+2 & -6-0 \\ -1-5 & 7+2 & -3-10 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 2 & -6 \\ -6 & 9 & 13 \end{pmatrix};
\end{aligned}
$$

And eg $A+C$: not defined since the matrices are different sizes.

Matrix Products

Some notation first: a *row matrix* is a matrix with 1 row; a *column matrix* is a matrix with 1 column.
To multiply 2 matrices **must have**: number of columns in the first matrix = number of rows in the second.

E.g.
The product of $A_{2\times 3} \times B_{3\times 2}$ can be calculated.
The product of $A_{2\times 3} \times C_{2\times 2}$ can *not* be calculated *but* $C_{2\times 2} \times A_{2\times 3}$ is defined.

Also and importantly: the order of multiplication matters, so $AB \neq BA$ in general. Another way of saying this is that *matrix multiplication is not commutative*.

definition of matrix multiplication:

Given, $A_{n\times k}, B_{k\times m}$ then $C = AB$ is an $n \times m$ matrix with entries

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$$

The entry at i, j is the sum of products of corresponding entries in the i^{th} row of A with the j^{th} column of B.

An example is the easiest way to see what is going on.
E.g
Consider 2 matrices

$$A = \begin{pmatrix} 1 & -3 & 0 \\ -2 & -1 & 4 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ 5 & 3 \\ -4 & -5 \end{pmatrix}$$

(check the sizes to verify they can be multiplied and $C = AB$ will be size 2×2)

$$
\begin{aligned}
C &= \begin{pmatrix} 1 & -3 & 0 \\ -2 & -1 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 5 & 3 \\ -4 & -5 \end{pmatrix} \\
&= \begin{pmatrix} 1(2)-3(5)+0(-4) & 1(-1)-3(3)+0(-5) \\ -2(2)-1(5)+4(-4) & -2(-1)-1(3)+4(-5) \end{pmatrix} \\
&= \begin{pmatrix} 2-15+0 & -1-9+0 \\ -4-5-16 & 2-3-20 \end{pmatrix} \\
&= \begin{pmatrix} -13 & -10 \\ -25 & -21 \end{pmatrix}
\end{aligned}
$$

Notice these numbers are not obvious given the original matrices! Now try BA: this is a 3×2 matrix by 2×3

Scalar multiplication

if c is a number (scalar) and A a matrix then cA is the matrix where every entry in A is multiplied by c.

E.g. if $c = 2$ and $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ then

$$cA = 2\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2(1) & 2(2) \\ 2(3) & 2(5) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 10 \end{pmatrix}$$

and

$$Ac = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} 2 = \begin{pmatrix} 1(2) & 2(2) \\ 3(2) & 5(2) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 10 \end{pmatrix} = cA.$$

Scalar multiplication is in general commutative. Also note that the matrix A can be any size.

Special matrices

Zero matrix

The zero matrix can be any size with all entries zero

Identity matrix

defined for square matrices only:

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ & \vdots & 0 & 1 \end{pmatrix}$$

ie one on the *main diagonal*, zero elsewhere. Check that $IA = AI = A$ for any square matrix A you try $\rightarrow$ I is the identity for matrix multiplication.

Some matrix properties

Transpose of A

If A is an $n \times m$ matrix then A^T is $m \times n$, obtained by swapping the rows and columns of A. A^T is called the *transpose* of A.

E.g.

$$A = \begin{pmatrix} 1 & 1 & 7 \\ 8 & 2 & 0 \end{pmatrix} \quad ; \quad A^T = \begin{pmatrix} 1 & 8 \\ 1 & 2 \\ 7 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 3 \\ 4 & 6 \end{pmatrix} \quad ; \quad B^T = \begin{pmatrix} -1 & 4 \\ 3 & 6 \end{pmatrix}$$

$$C = \begin{pmatrix} -12 & -7 \\ -7 & 10 \end{pmatrix} \quad ; \quad C^T = \begin{pmatrix} -12 & -7 \\ -7 & 10 \end{pmatrix} = C$$

In the last example $C^T = C$ and C is called a *symmetric matrix*.

properties of transpose

There are lots of nice properties of transpose including

- $\left(A^T\right)^T = A$
- $(AB)^T = B^T A^T$. This property can be proved as follows:
  $B^T A^T = (b_{ik})^T (a_{kj})^T = b_{ki} a_{jk} = a_{jk} b_{ki} = (AB)_{ji} = (AB)^T_{ij}$
- $(A+B)^T = A^T + B^T$.

Gauss-Jordan Elimination

Solving systems of linear equations

Consider

$$\begin{aligned} 3x - 2y &= 1 \\ -x + y &= 1 \end{aligned}$$

What x and y solve these simultaneous equations? Of course, you can easily find the answer with pen and paper ($x = 3, y = 4$)! But now, what if there were many more maybe thousands of unknowns and equations. This is much harder to solve with pen and paper and a systematic, algorithmic method that can easily be written as a computer programme for numerical solution is preferable.

Gauss-Jordan Elimination is such a method

Start by noting that the system of equations can be written as *a matrix equation*:

$$Ax = b$$

where A is the *matrix of coefficients*, x is the set (column matrix/vector) of unknowns and b is the column matrix formed from the numbers on the right-hand-side of the equations.

In our example above $A = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.