Linear Alg:

- essential props of mxs.
- how to model pop. growth + ...

Mx : a compact way of ~~recording~~ representing / storing info.

↳ a rectangular array of numbers ; denoted with cap. letters

if mx A has n rows, k cols. then its 'size' is $n \times k$.

eg $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ is a 2×3 mx

$A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \\ 2 & -3 \end{pmatrix}$ is a 3×2 mx.

if $n = k$ ie # rows = # cols then it is a sq. mx

eg $A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$ is 2×2.

To refer to a number in A (an entry) use its row, col position eg in mx A, entry $a_{\underset{\uparrow \text{row}}{2}\underset{\nwarrow \text{col.}}{3}}$ is found at row 2, col 3

eg if $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ then $a_{23} = 1$

Note, entries can be $\underset{\mathbb{Z}}{\text{integers}}$ (+, −), $\underset{\mathbb{R}}{\text{real numbers}}$ or $\underset{\mathbb{C}}{\text{Cmplx nos.}}$

Def^n $A_{n\times k}$, $B_{k\times m}$ then $C = AB$ is $n\times m$

and

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ik}b_{kj}$$

ie: entry at ij is the sum of prods of corr. entries in the i^{th} row of A with j^{th} col. of B

'strange & unnatural' but precisely the most useful defn

Eg $A = \begin{pmatrix} 1 & -3 & 0 \\ -2 & -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ 5 & 3 \\ -4 & -5 \end{pmatrix}$

Check $A \sim 2\times 3$, $B \sim 3\times 2$ $\Rightarrow$ so $C = AB = (2\times 2)$

$$C = \begin{pmatrix} 1 & -3 & 0 \\ -2 & -1 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 5 & 3 \\ -4 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 1(2)-3(5)+0(-4) & 1(-1)-3(3)+0(-5) \\ -2(2)-1(5)+4(-4) & -2(-1)-1(3)+4(-5) \end{pmatrix}$$

$$= \begin{pmatrix} 2-15+0 & -1-9+0 \\ -4-5-16 & 2-3-20 \end{pmatrix}$$

$$= \begin{pmatrix} -13 & -10 \\ -25 & -21 \end{pmatrix}$$

Eg: $L = \begin{pmatrix} 0.8 & 1.2 \\ 0.6 & 0.5 \end{pmatrix}$ with pop vec $v = \begin{pmatrix} 100 \\ 200 \end{pmatrix}$

Lv is $(2\times 2)(2\times 1) \rightarrow (2\times 1)_{mx}$

$$= \begin{pmatrix} 0.8 & 1.2 \\ 0.6 & 0.5 \end{pmatrix}\begin{pmatrix} 100 \\ 200 \end{pmatrix} = \quad 0.8(100)+1.$$

$$= \begin{pmatrix} 320 \\ 160 \end{pmatrix}$$

Eg

$$A_{2\times4} = \begin{pmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & -9 \end{pmatrix} ; \quad C_{4\times3} = \begin{pmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{pmatrix}$$

$AC \Rightarrow$ Size 2×3

$$= \begin{pmatrix} 1(8)-3(-3)+0(2)+4(-1) & 1(5)-3(10)+0+4(-7) & 1(3)-3(2)+0+4(5) \\ \vdots & & \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -53 & 17 \\ -56 & -23 & 81 \end{pmatrix}$$

Now

CA : X

$$(4\times\underbrace{3)\ (2}_{\#}\times4)$$

Note $B = \begin{pmatrix} 3 & -1 & 7 \\ 10 & 1 & -8 \\ 5 & 2 & 4 \end{pmatrix}$ $\quad D = \begin{pmatrix} -1 & 4 & 9 \\ 6 & 2 & -1 \\ 7 & 4 & 7 \end{pmatrix}$

both 3×3 so BD, DB 3×3

Check

$$BD = \begin{pmatrix} 40 & 38 & 77 \\ -60 & 10 & 33 \\ 45 & 0 & -19 \end{pmatrix}$$

$$DB = \begin{pmatrix} -8 & 23 & -3 \\ 43 & -6 & 22 \\ 26 & 11 & 45 \end{pmatrix}.$$

Matrix Add (Sub).

define sum/diff of two mx of same size. by adding (subtr) corresponding entries

Def^n. A, B $n\times m$ then $S = A + B$ also $n \times m$

and $s_{ij} = a_{ij} + b_{ij}$

then $D = A - B$ "

$d_{ij} = a_{ij} - b_{ij}$

Eg.

$$A = \begin{pmatrix} 3 & 0 & -6 \\ -1 & 7 & -3 \end{pmatrix}; \ B = \begin{pmatrix} 4 & -2 & 0 \\ 5 & -2 & 10 \end{pmatrix}; \ C = \begin{pmatrix} 3 & 9 \\ 2 & 1 \end{pmatrix}$$

Compute

$$\boxed{A+B} = \begin{pmatrix} 3 & 0 & -6 \\ -1 & 7 & -3 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 0 \\ 5 & -2 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 3+4 & 0-2 & -6+0 \\ -1+5 & 7-2 & -3+10 \end{pmatrix} = \begin{pmatrix} 7 & -2 & -6 \\ 4 & 5 & 7 \end{pmatrix}$$

$$\boxed{A-B} = \begin{pmatrix} 3 & 0 & -6 \\ -1 & 7 & -3 \end{pmatrix} - \begin{pmatrix} 4 & -2 & 0 \\ 5 & -2 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 3-4 & 0-(-2) & -6-0 \\ -1-5 & 7-(-2) & -3-10 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -6 \\ -6 & 9 & -13 \end{pmatrix}$$

$-A + AC$ → not same size ⇒ cannot add.

Matrix Products:

Notation: row mx → mx with 1 row
Col " → " " 1 col. > also called vectors

N.B: to multply 2 mx # cols in 1st $\doteq$ # row in 2nd

⇒ $A\ (2\times3) \times B\ (3\times2)$ ok ✓

$A\ (2\times3) \times C\ (2\times2)$ ✗ (but $C(2\times2) \times A(2\times3)$ ok ✓.)

AND ⇒ order of multiplication matters ie $AB \neq BA$.

Eg $A = \begin{pmatrix} 1 & 8 \\ 0 & 12 \end{pmatrix}$; $B = \begin{pmatrix} -3 & 4 \\ 7 & -10 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 8 \\ 0 & 12 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ 7 & -10 \end{pmatrix} = \begin{pmatrix} 1(-3)+8(7) & 1(4)+8(-10) \\ 0(3)+12(7) & 0(4)-12(10) \end{pmatrix}$$

$$= \begin{pmatrix} -3+56 & 4-80 \\ 0+84 & 0-120 \end{pmatrix}$$

$$= \begin{pmatrix} 53 & -76 \\ 84 & -120 \end{pmatrix}.$$

Scalar multiplication

if c is a number (scalar) ~~then~~ and A a matrix then cA = ~~c~~ mx where every entry in A is multiplied by c

eg $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ $c = 2$

$$\Rightarrow cA = \begin{pmatrix} 2(1) & 2(2) \\ 2(3) & 2(5) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 10 \end{pmatrix}$$

and

$$Ac = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}2 = \begin{pmatrix} 2(2) & 2(2) \\ 3(2) & 5(2) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 10 \end{pmatrix}$$

$\Rightarrow cA = Ac.$

Note A can be any size

Zero matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

describes growth of pops (and proj. age distribution) in pop ecology.

Note pop at yr N = P_N then

$$P_{N+1} = LA P_N$$
$$P_{N+2} = L L P_n$$
$$P_{N+3} = L L L P_N$$
$$\vdots$$
$$P_{N+t} = L^t P_N$$

Leslie matrices

Use matrices to store information on population growth. Call the proportion of the population hat survives from one season to the next the **survivability** and the number of offspring the **fecundity**.

A **Leslie matrix** square mx stores this information in a precise and well-defined way. So, for example considering a bird population, you can write down a Leslie matrix for the hatchlings and adults in this population along the lines

$$\begin{pmatrix} \text{average fecundity of hatchlings next year} & \text{average fecundity of adults next year} \\ \text{survivability of hatchlings} & \text{survivability of older birds} \end{pmatrix}$$

Consider a concrete example: The Leslie matrix for a bird population of 100 hatchlings and 200 adults is the following

$$L = \begin{pmatrix} 0.8 & 1.2 \\ 0.6 & 0.5 \end{pmatrix}$$

You can understand the entries in this matrix as

0.8: the average number of hatchlings next year from hatchlings this year. Since there are 100 hatchlings this year you should expect 0.8x100 = 80 hatchlings next year.

1.2: the average fecundity of adults is 1.2. Since there are 200 adults this year you should expect 1.2x200=240 hatchlings next year from this year's adult population.

0.6: the hatchling survivability is 0.6 ie. 60% of all hatchlings survive 'til next year. There are 100 hatchlings this year so 0.6x100=60 survive to become adults next year.

0.5: the adult survivability is 0.5. So, 50% of adults survive one year to the next. There are 200 this year so there will be 0.5x200=100 next year.

Use this information to estimate the number of hatchlings and adults next year.

PTO

Identity Matrix

Zero mx

$A_{n\times m}$ + $\begin{pmatrix}0&\\&\ddots\end{pmatrix}$

so $A_{n\times m} + 0_{n\times m}$

$= A_{n\times m}$

recall when multiplying numbers have

$$1(x) = x(1) = x \quad \forall x$$

is there an equivalent for mx?

→ identity mx for <u>square</u> <u>matrices</u> <u>only</u>

$$I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$ 1 on main diag. 0 elsewhere.

eg $I_2 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$; $I_4 = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

Check eg with $A = \begin{pmatrix}7&1\\3&2\end{pmatrix}$

$$AI = \begin{pmatrix}7&1\\3&2\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}7&1\\3&2\end{pmatrix} = A$$

$$IA = \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}7&1\\3&2\end{pmatrix} = \begin{pmatrix}7&1\\3&2\end{pmatrix} = A.$$

Transpose of A.

A^T. If A is an $n\times m$ mx, A^T is $m \times n$ mx obtained by swapping rows & cols of A.

eg $A = \begin{pmatrix}1&1&7\\8&2&0\end{pmatrix}$; $A^T = \begin{pmatrix}1&8\\1&2\\7&0\end{pmatrix}$

$A = \begin{pmatrix}-1&3\\4&6\end{pmatrix}$; $A^T = \begin{pmatrix}-1&4\\3&6\end{pmatrix}$.

(*) $A = \begin{pmatrix}-12&-7\\-7&10\end{pmatrix}$; $A^T = \begin{pmatrix}-12&-7\\-7&10\end{pmatrix} = A$

if $A^T = A$, A is a symmetric mx.

$$B^TA^T = (b^T)_{ik}(a^T)_{kj}$$
$$= b_{ki}a_{jk}$$
$$= a_{jk}b_{ki}$$
$$= (AB)_{ji}$$
$$= (AB)^T_{ij}$$

there are lots of props of transpose eg $(A^T)^T = A$

$(AB)^T = B^TA^T$, $\quad (A+B)^T = A^T + B^T$

⟹

Solving systems of Linear Eqⁿs.

Consider

$$3x - 2y = 1 \qquad \text{what } x, y \text{ solve this ?}$$
$$-x + y = 1$$

(by easy subs. $x = 3, y = 4$).

1 Note that this can be written as a mx eqⁿ

$$A x = b$$

where

A = mx of coeff
x = vector of unknowns
b = r.h.s

So eg

$$A = \begin{pmatrix} 3 & -2 \\ -1 & \cancel{g}1 \end{pmatrix}$$

$$x = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

does this work ?

$$A x = b$$

$$\Rightarrow \begin{pmatrix} 3 & -2 \\ -1 & \cancel{g}1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 3x - 2y = 1 \\ -x + y = 1 \end{array} \quad \checkmark$$

Now G-J-E :

i) ~~tran~~ - write the augmented mx ie $(A|b)$

ii) - transform A to Id. (also transforming b) by e.r.o.

iii) - answer is what appears where b was.

i) Augmented mx of our example $(A|b) = \left(\begin{array}{cc|c} 3 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right)$

ii) e.r.o,

1) a row can be switched with any other $R_1 \Leftrightarrow R_2$

$R_1 \to 3R_1$ 2) a " " " multiplied by any number (not 0)

$R_1 \to 2R_1 + R_2$ 3) a multiple of a row can be added to another row.

So $\left(\begin{array}{cc|c} 3 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ -1 & 1 & 1 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right)$

iii) now $A \to \mathbb{1}$ and ans is where b was

ie $\left| \begin{array}{l} x = 3 \\ y = 4 \end{array} \right|$

$\left(\begin{array}{cc|c} 3 & -2 & 1 \\ -1 & 1 & 1 \end{array}\right) \xrightarrow{3R_2 + R_1} \left(\begin{array}{cc|c} 3 & -2 & 1 \\ 0 & 1 & 4 \end{array}\right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{cc|c} 3 & 0 & 9 \\ 0 & 1 & 4 \end{array}\right)$

$\xrightarrow{R_1/3} \left(\begin{array}{ccc} 1 & 0 & \\ 0 & 1 & 4 \end{array}\right)$

Eg 2.

$$\begin{aligned} x + 2y &= 5 \\ 2x - y &= 0 \end{aligned}$$

$$\rightarrow \quad \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

then

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -10 \end{array}\right) \xrightarrow{R_2/-5} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

$x = 1,\ y = 2$: check $1 + 2(2) = 5$ ✓

$2(1) - 2 = 0$ ✓.

Eg 3

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 3 & 2 & 6 \\ 2 & 5 & 6 & 5 \end{array}\right) \xrightarrow{\substack{R_2 - R_1 \\ \cancel{R_1 + R_2}}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 2 & 5 & 6 & 5 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 1 & 0 & 3 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

$$\xrightarrow{R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{R_1 - 3R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

$$\begin{array}{ll} \text{I} & 2y + z = 4 \\ \text{II} & x + y + 2z = 6 \\ \text{III} & 2x + y + z = 7 \end{array} \qquad \left(\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 7 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right) \xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & -1 & -3 & -5 \end{array}\right)$$

$$\xrightarrow{2R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -5 & -6 \end{array}\right) \xrightarrow{-R_3/5} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 6/5 \end{array}\right)$$

[illegible]

$$\xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 0 & 14/5 \\ 0 & 0 & 1 & 6/5 \end{array}\right) \xrightarrow{R_2/2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 1 & 0 & 7/5 \\ 0 & 0 & 1 & 6/5 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 23/5 \\ 0 & 1 & 0 & 7/5 \\ 0 & 0 & 1 & 6/5 \end{array}\right) \xrightarrow{R_1 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 14/5 \\ 0 & 1 & 0 & 7/5 \\ 0 & 0 & 1 & 6/5 \end{array}\right)$$

(!)

$u \quad x = 11/5 \; ; \; y = 7/5 \; ; \; z = 6/5$

$\text{I} \quad 2(7/5) + \frac{6}{5} = \frac{20}{5} = 4$ ✓

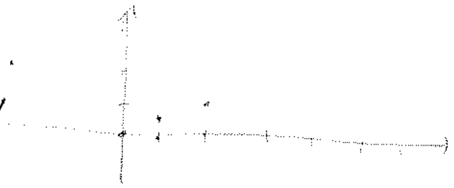
$\text{II} \quad \frac{71}{5} + \frac{7}{5} + 2\left(\frac{6}{5}\right) = \frac{30}{5} = 6$ ✓

$\text{III} \quad 2\left(\frac{11}{5}\right) + \frac{7}{5} + \frac{6}{5} = \frac{35}{5} = 7$ ✓

Eg $2x - 3y = 1$
$4x - 6y = 1$ $\rightarrow \left(\begin{array}{cc|c} 2 & -3 & 1 \\ 4 & -6 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 0 & -1 \end{array}\right)$?

$\xrightarrow{\cancel{2R_1}\ R_1/2} \left(\begin{array}{cc|c} 1 & -3/2 & 1/2 \\ 4 & -6 & 1 \end{array}\right)$

No solutions

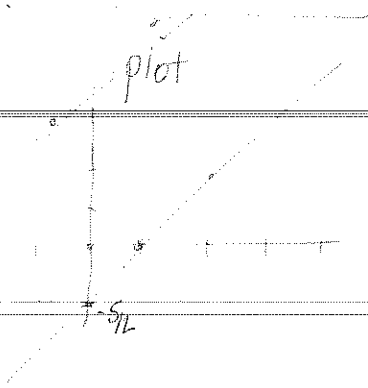


$\xrightarrow{R_2 - 4R_1} \begin{pmatrix} 1 & -3/2 & 1/2 \\ 0 & 0 & -1 \end{pmatrix}$!

$2x - y = -5$
$-2x + y = -2$

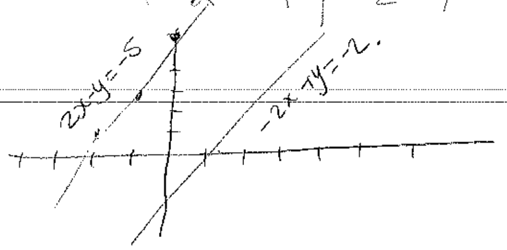
$\left(\begin{array}{cc|c} 2 & -1 & -5 \\ -2 & 1 & -2 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cc|c} 2 & -1 & -5 \\ 0 & 0 & -7 \end{array}\right)$?

plot



or

$\xrightarrow{R_1/2} \left(\begin{array}{cc|c} 1 & -1/2 & -5/2 \\ -2 & 1 & -2 \end{array}\right) \xrightarrow{R_1 + 1/2 R_2} \left(\begin{array}{cc|c} 0 & 0 & -7/2 \\ -2 & 1 & -2 \end{array}\right)$



general idea to fix A entries $\rightarrow \begin{pmatrix} d & c \\ a & b \end{pmatrix}$
but not necessary.

gaussian elimination $A \rightarrow \left(\begin{array}{cc|c} 1 & a & c \\ 0 & 1 & d \end{array}\right)$ ie upper triangular
giving $y = d$ from bottom row, x from $x + ay = c$ by subs.

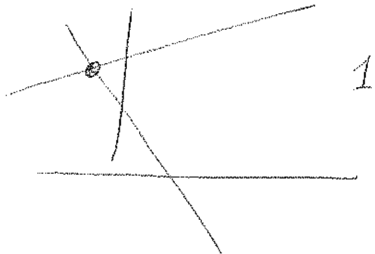
GJE : Complexity $\mathcal{O}(n^3)$, $n \times n$ matrix

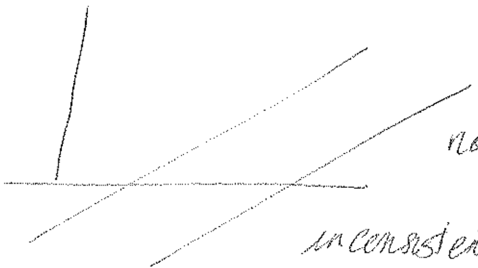
useful for medium-sized systems ie $\sim 1000 \times 1000$ mx.
↳ and gives exact ans.

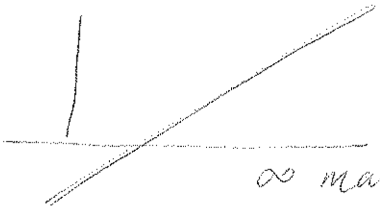
other methods for bigger systems (iterative but approximate)
↳ to machine prec.

Finishing sol^n of l eq^{n}s.

Note the possibilities for 2 eq^{n}s in 2 unknowns.







↳ in GJE

$\left|\begin{array}{cc|c} 2 & 2 & 2 \\ 4 & 4 & 4 \end{array}\right| \rightarrow \left(\begin{array}{cc|c} 2 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right)$

higher dims similar

ie 1 eq^n.

Matrix Inverse and Det.

Inv

A sq. mx. the inv. of A is A^{-1} s.th.

$$AA^{-1} = A^{-1}A = I.$$

if A^{-1} exists $\rightarrow$ A nonsingular

$A^{-1} \nexists \rightarrow$ A singular

Det :

how can you tell if A^{-1} exists (before trying to find it!)

define : the determinant of a 2x2 mx is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

eg $A = \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}$ has $\det(A) = 1(2) - 3(5)$

$= 2 - 15$

$= -13.$

For a 3x3 mx $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Theorem : A sq. mx is invertible if and only if its det. is not zero

eg $A = \begin{pmatrix} 3 & 7 \\ 9 & 2 \end{pmatrix} \Rightarrow \det(A) = 3(2) - 7(9) = -57 \neq 0$

$\Rightarrow A^{-1}$ exists

but what is it ?!

Cramer's Rule

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Eg $A = \begin{pmatrix} 5 & 3 \\ -4 & -2 \end{pmatrix} \Rightarrow \det A = (5)(-2) - (-4)(3) = 2$

$\Rightarrow A^{-1} = \frac{1}{2}\begin{pmatrix} -2 & -3 \\ 4 & 5 \end{pmatrix}$ and $\underbrace{\frac{1}{2}\begin{pmatrix} -2 & -3 \\ 4 & 5 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 5 & 3 \\ -4 & -2 \end{pmatrix}}_{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

So it works but gets cumbersome for large matrices

Better to use G-J-E.

<u>Method</u>:

i) given a sq. mx, eg $A = \begin{pmatrix} 5 & 3 \\ -4 & -2 \end{pmatrix}$

want A^{-1}, s. that $AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

ii) write augmented mx

$$(A \mid I) = \left(\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ -4 & -2 & 0 & 1 \end{array}\right)$$

iii) use ero's to transform A to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then I is A^{-1}

$$\cancel{\text{eg} \left(\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ -4 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_2/-4} \left(\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 1 & \frac{1}{2} & 0 & -\frac{1}{4} \end{array}\right)}$$

$$\cancel{\xrightarrow{5R_2 - 5R_1} \Big(5}$$

① eg $\left(\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ -4 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_1/5} \left(\begin{array}{cc|cc} 1 & \frac{3}{5} & 1/5 & 0 \\ -4 & -2 & 0 & 1 \end{array}\right)$

$$\xrightarrow{R_2 + 4R_1} \left(\begin{array}{cc|cc} 1 & 3/5 & 1/5 & 0 \\ 0 & 2/5 & 4/5 & 1 \end{array}\right)$$

$$\xrightarrow{5/2 R_2} \left(\begin{array}{cc|cc} 1 & 3/5 & 1/5 & 0 \\ 0 & 1 & 2 & 5/2 \end{array}\right)$$

$$\xrightarrow{R_1 - \frac{3}{5}R_2} \left(\begin{array}{cc|cc} 1 & 0 & -1 & -3/2 \\ 0 & 1 & 2 & 5/2 \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \quad \underbrace{\begin{matrix} -1 & -3/2 \\ 2 & 5/2 \end{matrix}}_{A^{-1}}$$

$A^{-1} = \begin{pmatrix} -1 & -3/2 \\ 2 & 5/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & -3 \\ 4 & 5 \end{pmatrix}$ as before

Eg 2. $A = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 3 & 12 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right)$$

no solution to G-J-E !

check $\det(A) = 1(12) - 4(3) = 0$

$\Rightarrow$ no A^{-1}.

Eg 3. $A = \begin{pmatrix} 3 & 7 \\ 9 & 2 \end{pmatrix}$ $\det A = -57$

A^{-1} exists ✓

$$\left(\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 9 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 0 & -19 & -3 & 1 \end{array}\right)$$

$$\xrightarrow{R_2/-19} \left(\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 0 & 1 & \frac{3}{19} & -\frac{1}{19} \end{array}\right)$$

$$\xrightarrow{R_1 - 7R_2} \left(\begin{array}{cc|cc} 3 & 0 & \overset{-2/19}{1 - \frac{21}{19}} & \overset{+7/19}{0 + \frac{7}{19}} \\ 0 & 1 & \frac{3}{19} & -\frac{1}{19} \end{array}\right)$$

$$\xrightarrow{R_1/3} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{2}{57} & +\frac{7}{57} \\ 0 & 1 & 3/19 & -1/19 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -2/57 & +7/57 \\ 3/19 & -1/19 \end{pmatrix} = \frac{1}{57}\begin{pmatrix} -2 & +7 \\ 9 & 3 \end{pmatrix}$$

$$= -\frac{1}{57}\begin{pmatrix} 2 & -7 \\ -9 & 3 \end{pmatrix}$$ à la Cramer.

$$A = \begin{pmatrix} 4 & 6 \\ 6 & -7 \end{pmatrix}$$

$$\xrightarrow{R_1/4} \left(\begin{array}{cc|cc} \cancel{4}1 & \cancel{6}3/2 & \cancel{1}/4 & 0 \\ 6 & -7 & 0 & 1 \end{array}\right) \xrightarrow{R_2 : R_2 - 6R_1} \left(\begin{array}{cc|cc} 1 & 3/2 & \frac{1}{4} & 0 \\ 0 & -16 & -3/2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2/-16} \left(\begin{array}{cc|cc} 1 & 3/2 & 1/4 & 0 \\ 0 & 1 & \frac{3}{32} & -\frac{1}{16} \end{array}\right) \xrightarrow{R_1 : R_1 - \frac{3}{2}R_2} \left(\begin{array}{cc|cc} 1 & 0 & 7/64 & \frac{3}{32} \\ 0 & 1 & 3/32 & -1/16 \end{array}\right)$$

i) $A^{-1} = \begin{pmatrix} 7/64 & 3/32 \\ 3/32 & -1/16 \end{pmatrix}$

$$= \frac{1}{64}\begin{pmatrix} 7 & 6 \\ 6 & -4 \end{pmatrix} = -\frac{1}{64}\begin{pmatrix} -7 & -6 \\ -6 & 4 \end{pmatrix}$$

Cramer's Rule result.

$$A = \begin{pmatrix} 3 & 7 \\ 9 & 2 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 9 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_1/3} \left(\begin{array}{cc|cc} 1 & 7/3 & 1/3 & 0 \\ 9 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - 9R_1} \left(\begin{array}{cc|cc} 1 & 7/3 & 1/3 & 0 \\ 0 & -19 & -3 & 1 \end{array}\right) \xrightarrow{R_2/-19} \left(\begin{array}{cc|cc} 1 & 7/3 & 1/3 & 0 \\ 0 & 1 & 3/19 & -\frac{1}{19} \end{array}\right)$$

$$\xrightarrow{R_1 - \frac{7}{3}R_2} \left(\begin{array}{cc|cc} 1 & 0 & -2/57 & 7/57 \\ 0 & 1 & 3/19 & -1/19 \end{array}\right)$$

ii $A^{-1} = \begin{pmatrix} -2/57 & 7/57 \\ 3/19 & -1/19 \end{pmatrix} = \begin{pmatrix} -2/57 & 7/57 \\ 9/57 & -3/57 \end{pmatrix}$

$$= -\frac{1}{57}\begin{pmatrix} 2 & -7 \\ -9 & 3 \end{pmatrix}$$ ✓

A nice application of A^{-1} to Sys. of Eq^{n}s.

Eg Solve
$$x + 3y = 14$$
$$2x - y = 49$$

ie in $Ax = b$ form

$$\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}14\\ 49\end{pmatrix}$$

$$A = \begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix} \Rightarrow \det(A) = 1(-1) - 3(2) = -7$$

$$\left(\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 2 & -1 & 0 & 1\end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 0 & -7 & 0-2 & 1\end{array}\right)$$

$$\xrightarrow{R_2/-7} \left(\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 0 & 1 & \frac{2}{7} & -\frac{1}{7}\end{array}\right)$$

$$\xrightarrow{R_1 - 3R_2} \left(\begin{array}{cc|c}1 & 0 & \\ 0 & 1 & \underbrace{\begin{matrix}\frac{1}{7} & +\frac{3}{7}\\ \frac{2}{7} & -\frac{1}{7}\end{matrix}}_{A^{-1}}\end{array}\right)$$

Now if
$$Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$Ix = A^{-1}b$$
$$\Rightarrow x = A^{-1}b = \frac{1}{7}\begin{pmatrix}1 & +3\\ 2 & -1\end{pmatrix}\begin{pmatrix}14\\ 49\end{pmatrix}$$
$$= \cancel{\begin{pmatrix}-19\\ 11\end{pmatrix}} \begin{pmatrix}23\\ -3\end{pmatrix}$$

check that $x = 23$, $y = -3$ works.

Now if I change b I don't need to recompute A^{-1}

say now want to solve

$$x + 3y = \cancel{7}\,1$$
$$2x - y = \cancel{56}\,4$$

have $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cancel{7}\,1 \\ \cancel{56}\,4 \end{pmatrix}$ ie $Ax = b$, $b = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

so

$$\underbrace{\frac{1}{7}\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}}_{A^{-1}A = I}\begin{pmatrix} x \\ y \end{pmatrix} = \cancel{\begin{pmatrix} 7 \\ 56 \end{pmatrix}} \quad \frac{1}{7}\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \cancel{7}\,1 \\ \cancel{56}\,4 \end{pmatrix}$$

ie $A^{-1}Ax = A^{-1}b$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \cancel{7}\,1 \\ \cancel{56}\,4 \end{pmatrix}$$

$$= \cancel{\frac{1}{7}\begin{pmatrix} 25 \\ -6 \end{pmatrix}} \quad \frac{1}{7}\begin{pmatrix} 13 \\ -2 \end{pmatrix}$$

check $x = \cancel{\frac{25}{7}}\,\frac{13}{7}$ $y = -\cancel{\frac{6}{7}}\,\frac{2}{7}$ solves eq^{ns}

$\frac{13}{7} + 3\left(-\frac{2}{7}\right) = 1$ ✓ ~~$\frac{25}{2} + 3\left(-\frac{6}{2}\right) = 7$~~ ✓

$2\left(\frac{13}{7}\right) - \left(-\frac{2}{7}\right) = \frac{28}{7} = 4$ ✓ ~~$2(25) + 6 = 56$~~ ✓

So with many systems that have same A ie

$Ax_1 = b_1$, $Ax_2 = b_2$, $Ax_3 = b_3$ . . . .

use A^{-1} to find x_i

rties of inverse :

niqueness : if $\exists A^{-1}$, it is unique $(\exists ! A^{-1})$

$B)^{-1} = B^{-1}A^{-1} \longrightarrow$ Proof: Consider $(AB)(B^{-1}A^{-1})$

$= ABB^{-1}A^{-1} = A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1} = \cancel{A}I$

Similarly $(B^{-1}A^{-1})(AB) = I$

$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

$I^{-1} = I \quad \forall\ n\times n.$

$^{-1})^{-1} = A$

Proof: $AA^{-1} = I$

$(AA^{-1})^{-1} = I^{-1}$

$(A^{-1})^{-1}A^{-1} = I$

$(A^{-1})^{-1}\underbrace{A^{-1}A}_{I} = IA = A$

$(A^{-1})^{-1} = A.$

E'values & E'vecs

derive info. about mx
↳ in particular *dynamic* info or info. about change of a system. [Solving $Ax=b$ just steady state]

app. quantify pop. grwth rate +.

Recall a vec. is just a row, col. mx. We will consider col. vecs that is of size $n \times 1$.

Eg $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$

Now consider mx-vec mult. for $u = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$Au = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \end{pmatrix} \leftarrow \text{nothing special}$$

$$Av = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2v$$

---

ie $$\underset{\substack{\uparrow \\ \text{e'vec}}}{A\,v} = \underset{\substack{\uparrow \\ \text{e'value} \sim \lambda}}{2}v$$

How do we find e'values, e'vecs given A?

Thm let A be an $n \times n$ mx. If λ is an e'value of A then $\det(A - \lambda \mathbb{1}) = 0$.

$\det(A - \lambda I) = 0$ is the characteristic eq^n of A.

use it to find λ.

Eg find e'values of $A = \begin{pmatrix} -7 & 18 \\ -3 & 8 \end{pmatrix}$.

well,
$$A - \lambda I = \begin{pmatrix} -7 & 18 \\ -3 & 8 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -7-\lambda & 18 \\ -3 & 8-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = \det\begin{pmatrix} -7-\lambda & 18 \\ -3 & 8-\lambda \end{pmatrix}$$

$$= (-7-\lambda)(8-\lambda) - 18(-3)$$

$$= -56 + 7\lambda - 8\lambda + \lambda^2 + 54$$

$$= \lambda^2 - \lambda - 2$$

and $\det(A-\lambda I) = 0$

$$\Rightarrow \lambda^2 - \lambda - 2 = 0$$

$$\Rightarrow (\lambda-2)(\lambda+1) = 0 \Rightarrow \boxed{\lambda = 2,\ \lambda = -1}$$

Note 2x2 mx will have 2 e'values from solving a quadratic

a 3x3 " " " 3 "

" " cubic etc.

Now to find e'vectors:

Each λ satisfies $Av = \lambda v^{(*)}$ for a particular v

Find v from (*)

Eg above, $\lambda = 2$

then want the v. s.th. $Av = 2v$ or $(A-2I)v=$

ie $\begin{pmatrix} -7 & 18 \\ -3 & 8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\Rightarrow \begin{array}{lcl} -7v_1 + 18v_2 = 2v_1 & \Rightarrow & 18v_2 = 9v_1 \text{ ie } 2v_2 = v_1 \\ -3v_1 + 8v_2 = 2v_2 & \Rightarrow & 68v_2 = 3v_1 \text{ ie } 2v_2 = v_1 \end{array}$$

so any 2 numbers s.th. $2v_2 = v_1$ will work.

eg if $v_2 = 1$ then $v_1 = 2 \Rightarrow v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

is an e'vec.

Check that $Av = \lambda v$ for $\lambda = 2$, $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Now repeat for $\lambda = -1$.

$$\begin{pmatrix} -7 & 18 \\ -3 & 8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Rightarrow \begin{array}{lcl} -7v_1 + 18v_2 = -v_1 & \Rightarrow & 18v_2 = 6v_1 \\ -3v_1 + 8v_2 = -v_2 & & 9v_2 = 3v_1 \end{array}$$

$$\Rightarrow v_1 = 3v_2$$

so $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ an e'vec. for $\lambda = -1$.

Note in general we solve $(A - \lambda I)v = 0$ by G-J.E.

eg $\lambda = 2$:

$$\left(\begin{pmatrix} -7 & 18 \\ -3 & 8 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -9 & 18 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

---

so $\left(\begin{array}{cc|c} \underbrace{-9 \quad 18}_{} & & 0 \\ -3 & 6 & 0 \end{array}\right) \xrightarrow{\text{by ero transf } A \to I} \cdots$

$$\underbrace{\begin{matrix} -3 & 6 \end{matrix}}_{\text{to } I}$$

Check that $-\frac{1}{9}R_1$, $R_2 + 3R_1$ gives

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \text{ ie } \boxed{v_1 = 2v_2}$$

1 more eg:

Given $A = \begin{pmatrix} 11 & 20 \\ -6 & -11 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 11 & 20 \\ -6 & -11 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 11-\lambda & 20 \\ -6 & -11-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (11-\lambda)(-11-\lambda) - 20(-6) = 0$$

$$\Rightarrow -121 + \lambda^2 + \overset{120}{\cancel{180}} = 0$$

$$\lambda^2 = \cancel{95}\ 1 \Rightarrow \lambda = \pm \cancel{9.75}\ 1$$

$\lambda = \cancel{9.75}\ 1$ solve $Av = \lambda v$

$$\begin{pmatrix} 11 & 20 \\ -6 & -11 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cancel{9.75}v_1 \\ \cancel{9.75}v_2 \end{pmatrix}$$

~~$v_2 = -0.0625 v_1$~~

~~$11v_1 + 20v_2 = 9.75v_1 \Rightarrow 20v_2 = -1.25 v_1$~~

~~$-6v_1 - 11v_2 = 9.75v_2 \qquad -20.75v_2 = 6v_1$~~

$$\begin{pmatrix} 11v_1 + 20v_2 = v_1 \\ -6v_1 - 11v_2 = v_2 \end{pmatrix} \Rightarrow \begin{matrix} 20v_2 = -10v_1 \Rightarrow v_2 = -\frac{1}{2}v_1 \\ -12v_2 = 6v_1 \qquad v_2 = -\frac{1}{2}v_1 \end{matrix}$$

So $v = \begin{pmatrix} v_1 \\ -v_1/2 \end{pmatrix} = t\begin{pmatrix} 10 \\ -5 \end{pmatrix}$

$\lambda = -1$

$$11v_1 + 20v_2 = -v_1 \Rightarrow 20v_2 = -12v_1 \Rightarrow v_2 = -\frac{3}{5}v_1$$

$$-6v_1 - 11v_2 = -v_2 \qquad -10v_2 = 6v_1 \Rightarrow v_2 = -\frac{3}{5}v_1$$

$$v = \begin{pmatrix} v_1 \\ -3/5 v_1 \end{pmatrix} = t\begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

e'vec.
$v_1 = (3+2i)v_2$

Eg with Cmplx e'values:

$$A = \begin{pmatrix} 6 & -13 \\ 1 & 0 \end{pmatrix} \Rightarrow \det\begin{pmatrix} 6-\lambda & -13 \\ 1 & 0-\lambda \end{pmatrix} = 0$$

$$-\lambda(6-\lambda) + 13 = 0$$

$$\lambda^2 - 6\lambda + 13 = 0 \quad = \quad \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2} = \frac{6 \pm 4i}{2}$$

$$= \underline{\underline{3 \pm 2i}}$$

the checkerboard mx

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

has

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{pmatrix}$$

$$= 1-\lambda \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & 1-\lambda \\ 1 & -1 \end{vmatrix}$$

$$= 1-\lambda\left[(1-\lambda)(1-\lambda)-1\right] + (-)(1-\lambda) + 1 + 1\left(1-(1-\lambda)\right)$$

$$= 1-\lambda\left[(1-\lambda)(1-\lambda)-1\right] - 1+\lambda+1+1-1+\lambda$$

$$= (1-\lambda)^3 - 1+\lambda -1+\lambda +1+1-1+\lambda$$

$$= (1-\lambda)^3 \quad -1+3\lambda$$

$$= 1-\lambda\left(1-2\lambda+\lambda^2\right) - 1+3\lambda$$

$$= 1-2\lambda+\lambda^2-\lambda+2\lambda^2-\lambda^3-1+3\lambda$$

$$= -\lambda^3+3\lambda^2$$

$$= -\lambda^2(\lambda-3).$$

$$\lambda = 0, \quad \lambda = 3.$$

λ occurs one → simple e'value

λ " $k \geq 2$ times → e'val. multiplicity k.

Properties:

- if $\lambda = 0$ an e'val $\Rightarrow$ A not invertible
- A, A^T same e'vals.
- dominant/principal e'vec corresponds to largest mag. $(|\lambda|)$ e'value.
- $\lambda \in \mathbb{R}$ or $\mathbb{C}$

λ occurs once $\rightarrow$ simple e'val.

λ " $k \geq 2$ times $\rightarrow$ e'val with mult. k.

E'values & Growth.

Can show that

For a Leslie mx with largest e'value λ_1. The rel. growth rate approaches λ_1 as the # years appr. ∞.

Thus the largest e'value of $\mathcal{L}$ is the <u>long-term growth rate</u>

and l-t % gr. rate $= (1-\lambda_1)\%$. $(\lambda_1 - 1)\%$.

<u>Note</u>:

population $= \Sigma$ entries of pop vec.

<u>define</u>

<u>rel. growth rate</u> as ratio of pop. 1 year to prev. yr.

eg. if pop grows from 100 to 150

then rel. growth rate $\frac{150}{100} = 1.5$ ie 50% growth rate.

For $\mathcal{L} = \begin{pmatrix} 0.5 & 2 \\ 0.5 & 0.5 \end{pmatrix}$

$\lambda_1 > 1$ growth

$\lambda_1 = 1$ equilibrium.

$\lambda_1 < 1$ extinction.

$$(0.5-\lambda)(0.5-\lambda) - 2(0.5) = 0$$

$$0.25 - \lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - \lambda - \tfrac{3}{4} = 0 \qquad (\lambda + \tfrac{1}{2})(\lambda - \tfrac{3}{2})$$

$$\Rightarrow \lambda = -\tfrac{1}{2} ; \boxed{\lambda = \tfrac{3}{2}}$$

$\lambda v_1 = \lambda v_2$ Corresponding e'vec $\longleftarrow$ ie l.t.g.r. $= 1.5$

is stable age pop. (as shown)

% g.r. $= 50\%$.

Calculating A^n, n large.

$\longrightarrow$ why? recall : $P_{N+t} = L^t P_N$

$\uparrow$ pop after t yrs

t might be 50 ....

find e'val. + e'vecs say $\lambda_1, \lambda_2, \lambda_3$ and V_1, V_2, V_3

write P with V_1 1st col, V_2 2nd col. etc.

Can show $A = PDP^{-1}$ $\quad D = \text{diag}(\lambda_1 .. \lambda_3)$

$$A^2 = PD^2P^{-1}$$

$$A^4 = PD^4P^{-1} \ \ldots$$

reading wk. exercise

does it work?

take $A = \begin{pmatrix} -7 & 18 \\ -3 & 8 \end{pmatrix}$, $\lambda_1 = 2$, $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = -1$ $v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

so $P = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ $\longrightarrow$ find P^{-1} by Cramer's rule

ie $\det(P) = 2(1) - 3(1) = -1$

$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ $\Rightarrow P^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

Now

$$A = PDP^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 6 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -7 & 18 \\ -3 & 8 \end{pmatrix} \checkmark$$

---

so eg

$$A^4 = PD^4P^{-1}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^4 & 0 \\ 0 & -1^4 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -29 & 90 \\ -15 & 46 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$ check.

$$A^{10} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{10} & 0 \\ 0 & -1^{10} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -2045 & 6138 \\ -1023 & 3070 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1024 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$$ check?!

Computation of mx pwrs by PD^nP^{-1} req. $\cancel{\frac{n^2(n-1)}{2}}$ adds + multip.

ie: flops

~~ie about~~ $\frac{1}{2}$

GET / HTTP/1.0

Calculating Fibonacci Numbers.

Consider a sequence $\{a_n\}_{n\geq 1}$ defined by

$$a_0 = 0, \quad a_1 = 1 \text{ and } a_{k+1} + a_k, \quad \forall k \geq 1$$

i.e. 0, 1, 1, 2, 3, 5, ...

Say you want to know a_{100}, a_{106}?

– lin. alg.: $a_{k+2} = a_{k+1} + a_k$

or $u_k = \begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix}, \quad k \geq 0$

then $u_{k+1} = \begin{pmatrix} a_{k+2} \\ a_{k+1} \end{pmatrix} = \begin{pmatrix} a_{k+1} + a_k \\ a_{k+1} \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix}$$

So if $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

then

$$u_{k+1} = A u_k \quad \underline{\text{and}} \quad u_{k+1} = A^k \underset{\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\underset{\downarrow}{u_0}}$$

gives a_{k+1} for any k if A^k known