

Inhomogeneous 2nd order

Returning to the inhomogeneous form $ay''(x) + by'(x) + cy(x) = f(x)$ we still require a particular integral y_p , that is *one* solution of the full equation. If f is exponential then it is straightforward: y_p is an exponential proportional to f .

- **For example:** consider

$$y'' + 3y' + 2y = e^x. \quad (1)$$

Trying $y_p = Ce^x$ where C is a constant gives

$$(1 + 3 + 2)Ce^x = e^x \quad (2)$$

so

$$C = \frac{1}{6} \quad (3)$$

and, therefore,

$$y_p(x) = \frac{1}{6}e^x \quad (4)$$

is a particular integral. To obtain the general solution we require the general solution of the homogeneous equation, see earlier example, $y_c = C_1e^{-x} + C_2e^{-2x}$. Thus, the general solution is

$$y(x) = y_c + y_p = C_1e^{-x} + C_2e^{-2x} + \frac{1}{6}e^x \quad (5)$$

If f is a solution of the homogeneous ODE this doesn't work.

- **For example:** consider $y'' + 3y' + 2y = e^{-x}$. Trying $y_p = Ce^{-x}$ gives $C = \infty$. Much as in the equal-root case for the homogeneous equation try $y_p = Cxe^{-x}$. Differentiating

$$\begin{aligned} y_p' &= Ce^{-x} - Cxe^{-x} \\ y_p'' &= -2Ce^{-x} + Cxe^{-x} \end{aligned} \quad (6)$$

Inserting these into the ODE gives

$$Cxe^{-x}(1 - 3 + 2) + Ce^{-x}(-2 + 3) = e^{-x} \quad (7)$$

so that $C = 1$ or $y_p = xe^{-x}$. The general solution can be written

$$y = (C_1 + x)e^{-x} + C_2e^{-2x} \quad (8)$$

¹Conor Houghton, houghton@maths.tcd.ie, see also <http://www.maths.tcd.ie/~houghton/231>

²Based on notes I got from Chris Ford

Now, for more general cases, we exploit the linearity. If f is a sum of exponentials, as, for example, $\sin x = (e^{ix} - e^{-ix})/(2i)$, just add up the corresponding particular integrals corresponding to each exponential term; so, for example, if

$$y'' + 3y' + 2y = \sin x. \quad (9)$$

We solve

$$y'' + 3y' + 2y = \frac{1}{2i}e^{ix} \quad (10)$$

and

$$y'' + 3y' + 2y = -\frac{1}{2i}e^{-ix} \quad (11)$$

and add the two solutions.

If f is not a finite sum of exponentials decompose f into complex exponentials: we use Fourier analysis.

- **Example with Fourier analysis:** Obtain the general solution of

$$y''(x) + 3y'(x) + 2y(x) = f(x) \quad (12)$$

where f is the periodic square wave

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases} \quad (13)$$

with $f(x + 2\pi) = f(x)$. The complementary function is $y_c = C_1e^{-x} + C_2e^{-2x}$, as in earlier example. For the particular integral we expand f as a Fourier series, this we have done before;

$$f(x) = \frac{4}{\pi} \sum_{n \text{ odd} > 0} \frac{1}{n} \sin nx = \frac{2}{\pi i} \sum_{n \text{ odd} \in \mathbb{Z}} \frac{1}{n} e^{inx} \quad (14)$$

Now, we find the particular integral for

$$y'' + 3y' + 2y = e^{inx} \quad (15)$$

Trying $y_p = Ce^{inx}$ gives

$$C(-n^2 + 3in + 2)e^{inx} = e^{inx} \quad (16)$$

so that

$$C = \frac{1}{-n^2 + 3in + 2} \quad (17)$$

Now, adding the individual particular integrals together to get the particular integral of full problem

$$y_p(x) = \frac{2}{\pi i} \sum_{n \text{ odd} \in \mathbb{Z}} \frac{e^{inx}}{n(-n^2 + 3in + 2)} \quad (18)$$

If f is not periodic write it as a Fourier integral

- **Fourier integral example:** Find the general solution of the ODE

$$y''(x) + 2y'(x) + 2y(x) = f(x) \quad (19)$$

where $f(x) = e^{-x^2}$. The auxiliary equation is $\lambda^2 + 2\lambda + 2 = 0$ with two complex roots $\lambda = -1 \pm i$ so that

$$y_c = e^{-x}(A \cos x + B \sin x) \quad (20)$$

Write f as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk e^{ikx} \tilde{f}(k) \quad (21)$$

where, using the standard formula for the Fourier integral of a Gaussian,

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ikx} e^{-x^2} = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}k^2} \quad (22)$$

Now, we obtain PI for $y'' + 2y' + 2y = e^{ikx}$. Trying $y = Ce^{ikx}$ gives

$$C(-k^2 + 2ik + 2)e^{ikx} = e^{ikx} \quad (23)$$

giving

$$C = \frac{1}{-k^2 + 2ik + 2}. \quad (24)$$

The particular integral of the whole problem is obtained by integrating over the particular integrals for the individual problems labelled by k :

$$y_p(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dk \frac{e^{ikx} e^{-k^2/4} (-k^2 + 2ik + 2)}{(-k^2 + 2ik + 2)}. \quad (25)$$