Further material for lectures: week of March 23rd

Higher-order ODEs

The numerical methods we are discussing can be generalised to solve higher-order initial value problems i.e. ODEs involving second derivatives and higher.
Problems involving n^{th}-order ODEs can always be reduced to n **coupled first-order ODEs**.
A general, nth-order differential equation in a single variable, $x(t)$ can be written as

$$\frac{d^n x}{dt^n} = f\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right). \tag{1}$$

Generalising our definition of an **initial value problem**, the values of x and all its derivatives up to $\frac{dx^{n-1}}{dt^{n-1}}$ are specified for some value of $t = t_0$.

Rewriting an nth order ODE

The method is illustrated with an example. Consider the second-order inhomogeneous ODE

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x). \tag{2}$$

Define a new variable $z(x)$ as

$$z(x) = \frac{dy}{dx}, \tag{3}$$

and substitute this in the original ODE, Eqn. 2, to get

$$\frac{dz}{dx} + q(x)z(x) = r(x). \tag{4}$$

In this way, we have transformed the single second-order ODE given by Eqn. 2 to **two coupled first-order ODEs**.

$$\begin{aligned} \frac{dy}{dx} &= z(x), \\ \frac{dz}{dx} &= r(x) - q(x)z(x). \end{aligned}$$

Note that the coupled equations are written here in the same form as equations that can be solved by numerical methods e.g. the Euler method. Also recall that the Euler (and higher-order) numerical methods solve initial value problems which means that for each of the coupled equations we need a separate initial condition. In this example, this means we need two initial conditions - one for y and one for z (which of course corresponds to an initial condition for the derivative of y) for example

$$y(0) = \alpha, \quad \frac{dy(0)}{dx} = z(0) = \beta. \tag{5}$$

Given this information we can use the Euler or Runge-Kutta numerical methods to find an approximate solution. In this case, with Euler, we have two numerical solutions to evolve from their respective initial conditions, as

$$\begin{aligned} y_{i+1} &= y_i + h z_i, && (6) \\ z_{i+1} &= z_i + h\left(r(x_i) - q(x_i) z_i\right). && (7) \end{aligned}$$

Why are these coupled equations? To see this note that for each step the value of y, determined from Eqn. 6 depends on the value of z at the previous step, so both equations must be iterated together.

Harmonic oscillator with friction

Consider a spring, stretched a distance x_0 and then released. The position thereafter is described by a second-order differential equation (for damped harmonic motion)

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x(t) = 0, \tag{8}$$

where β and ω are constants, and for this example we will use $\beta = 0.1$ and $\omega^2 = 1$.

Can we calculate the position x and some time t after release, using numerical techniques?

We reduce the second-order equation to two coupled first-order equations and use the Euler method - choosing $h = 0.1$, as follows. Write

$$\frac{dx}{dt} = p(t) \tag{9}$$

and substitute in Eqn. 8, yielding

$$\frac{dp}{dt} + 2\beta p(t) + \omega^2 x(t) = 0. \tag{10}$$

Therefore the two coupled equations to solve are

$$\begin{aligned} \frac{dx}{dt} &= p(t) \quad \text{solving this gives } x(t) \\ \frac{dp}{dt} &= -2\beta p(t) - \omega^2 x(t) \quad \text{solving this gives } p(t). \end{aligned}$$

And the initial conditions are

$$\begin{aligned} \text{velocity}: \quad & \frac{dx}{dt} = p(0) = 0, \\ \text{position}: \quad & x(0) = 1. \end{aligned}$$

We can now numerically solve these equations.

$t = 0$

$$x_0 = 1 \quad p_0 = 0$$

$t = 0.1$

First equation

$$\frac{dx}{dt} = p(t)$$

$$\begin{aligned} Euler: \quad x_1 = \quad & x_0 + hf(x_0, t_0) \\ x_1 = \quad & 1 + 0.1p_0 \\ x_1 = \quad & 1 \end{aligned}$$

Second equation

$$\frac{dp}{dt} = -2\beta p(t) - \omega^2 x(t)$$

$$\begin{aligned} Euler: \quad p_1 = \quad & p_0 + hg(p_0, x_0, t_0) \\ p_1 = \quad & 0 + 0.1[-0.2p_0 - 1x_0) \\ p_1 = \quad & 0 + 0.1[0 - 1] \\ p_1 = \quad & -0.1 \end{aligned}$$

Only now we can solve for

$t = 0.2$

$$\begin{aligned} Euler: \quad x_2 = \quad & x_1 + hf(x_1, t_1) \\ x_2 = \quad & 1 + 0.1p_1 \\ x_2 = \quad & 1 + 0.1(-0.1) \\ x_1 = \quad & 0.99 \end{aligned}$$

Therefore each step requires solving 2 ODES.

$$\begin{aligned} Euler: \quad p_2 = \quad & p_1 + hg(p_1, x_1, t_1) \\ p_2 = \quad & -0.1 + 0.1[-0.2p_1 - 1x_1) \\ p_2 = \quad & 0 + 0.1[-0.2(-0.1) - 1(1)] \\ p_2 = \quad & -0.198 \end{aligned}$$

$t = 0.3$

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Note that we solved this equation analytically using the auxilary equation method in earlier lectures in this module. Figure 1 shows the solution to this second-order ODE using the parameter values specified and implementing the Euler method on an interval $t = 0$ to $t = 50$ with a step-size $h = 0.001$.

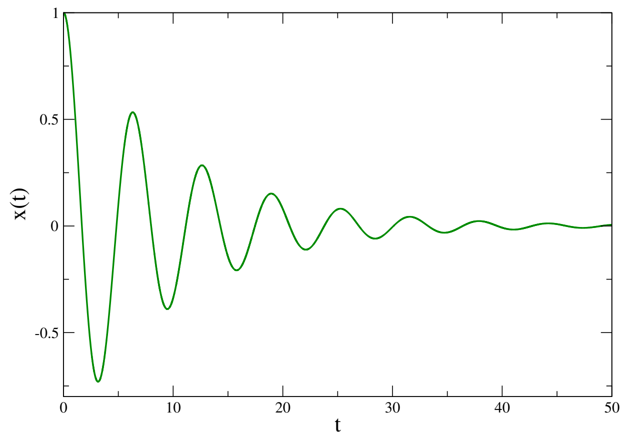

Figure 1: Plot of position as a function of time, determined using the Euler method with $\beta = 0.1$, $\omega^2 = 1$ and $h = 0.001$. Some simple code in C to implement this is included as a downloadable file with these notes.

For the same parameters used here you can compare your numerical solution at $x(t)$ with the exact solution. The modified Euler (2nd order Runge-Kutta) or the 4th order Runge-Kutta will yield more precise results. Implementing e.g. the 4th order RK will require 4 evaluations of $f(x, t)$ for each equation that you solve.

Aside

As noted in passing when we solved this same equation analytically, using the auxilary equation method, the answer x at each time t will depend on the values of β and ω - in the auxilary equation method the values of β and ω determine the form of the roots of the auxilary equation: real, degenerate, complex etc. There are in fact four cases:

- $\beta^2 = \omega^2$: critical damping,
- $\beta^2 > \omega^2$: over damping,
- $\beta^2 < \omega^2$: under damping,
- $\beta = 0$: no damping.

In the worked example here $\beta = 0.1 \Rightarrow \beta^2 = 0.01$ and $\omega^2 = 1$ so this is an under damped system.