Material intended for lectures March 18th and 20th

Euler method

A short summary of the Euler method, covered in lectures already.

Given the IVP $y'(t) = f(t, y)$ with initial condition $y(t_0) = y_0$, we want to evaluate (solve for) $y(t_n)$. The interval (t_0, t_n) is divided into n steps of size $h = |t_n - t_0|/n$ and $y(t_k)$ is evaluated in sequence

$$y(t_k) = y(t_{k-1}) + hf(t_{k-1}, y(t_{k-1})),\ k = 1, \ldots, n. \tag{1}$$

Euler's method is easy to understand and to implement numerically but is not often used in practice since:

- it's not accurate (we will see below that the error is linear in stepsize)
- it's not efficient (there are better schemes ie better accuracy for similar computing cost)
- it is not stable - for many ODEs the Euler approximate solution diverges quickly from the true solution.

Figure 1 shows the numerical solution of the IVP for the ODE $\frac{dx}{dt} = x(t)$ with $x(0) = 1$.

Euler Method and Error Analysis

Consider the solution of the IVP $y'(t) = f(t, y)$ with $y(t_0) = y_0$, denoted $\phi(t)$.

- the Euler formula $y_{n+1} = y_n + hf(t_n, y_n)$ approximates the solution, with $y_n \approx \phi(t_n)$.
- We saw that emperically we can expect the error to decrease as the step-size h decreases.
- A natural question then is: how small does h have to be for a certain tolerance?

To answer this question we need to understand the different errors in our numerical solution of the ODE. These errors can be classified as

- Local truncation error: e_n the amount of error at each step
- Global truncation error: E_n the error in the solution ie the difference between the algorithm and $\phi(t)$.
- Round-off error: R_n the error due to the finite precision available on a computer to represent numbers.

We will discuss the local and global trucation errors here and take up the discussion on round-off error later, time permitting.

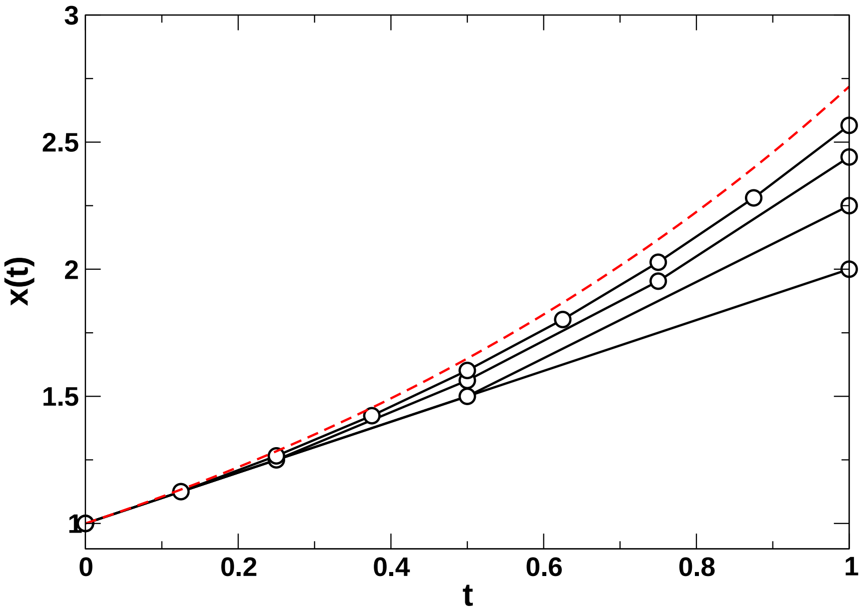

Figure 1: The solution of the ODE $x' = x, x(0) = 1$ for different step-sizes, on the interval $(0, 1)$. The exact solution is shown with the dashed red line.

Local Truncation Error

Assume that $\phi(t)$ solves the IVP so that

$$\phi'(t) = f(t, \phi(t)). \tag{2}$$

Taylor's theorem (via the mean value theorem) allows us to evaluate ϕ by a near-by value with an accuracy that depends on the derivatives of ϕ.

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \frac{1}{2!}\phi''(\tilde{t_n})h^2 + \dots \tag{3}$$

and $\tilde{t_n} \in (t_n, t_n + h)$. The expression is exact if all terms in the Taylor expansion on the RHS are kept. Keeping terms up to the second derivative we see that

$$\phi(t_n + h) \approx \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(\tilde{t_n})h^2, \tag{4}$$

and since ϕ is a solution of the IVP we can write

$$\phi(t_n + h) \approx \phi(t_n) + hf(t_n, \phi(t_n)) + \frac{1}{2}\phi''(\tilde{t_n})h^2. \tag{5}$$

Now, in moving one step (size h) from t_n to t_{n+1} the difference between the "true" solution $\phi(t_{n+1})$ and the euler approximate solution y_{n+1} is

$$e_{n+1} = \phi(t_{n+1}) - y_{n+1} = \frac{1}{2}\phi''(\tilde{t_n})h^2. \tag{6}$$

Now, we assume there is a uniform bound $M = \max_{t\in[a,b]}|\phi''(t)|$ and then the local error is bounded by

$$|e_n| \leq \frac{Mh^2}{2} = Kh^2, \tag{7}$$

where K is a constant.

I didn't mention in the earlier lines, but we have also assumed the third derivative is bounded on the interval we are interested in (so that neglecting that term and higher derivatives is reasonable).

The Euler mehtod has a local trucation error of order h^2, which is often written as $\mathcal{O}(h^2)$.

This expression for the local error, in terms of a parameter that we control ie the step size h, allows us to choose the step-size (h) to keep the numerical solution within a certain tolerance ϵ at each step. We can write

$$\frac{Mh^2}{2} \leq \epsilon \;\text{ or }\; h \leq \sqrt{\frac{2\epsilon}{M}}. \tag{8}$$

In practice it is often difficult to estimate the second-derivate term $|\phi''|$ (or M).

0.1 Global Truncation Error

Remember that we use Euler to solve and IVP problem which means we start from an initial (known) value and we want to know the value/solution at some later point (or time). We get there by dividing the interval into small steps and evaluating the solution after each step using the value from the previous step. So, we have control over the step-size h and if the interval is e.g.

$$t_{\text{start}} - t_{\text{finish}}, \tag{9}$$

and the step size is h then the number of steps taken will be

$$n = \frac{|t_{\text{start}} - t_{\text{finish}}|}{h}. \tag{10}$$

Each step introduces an error and after n steps

$$\begin{aligned} E_n &= \frac{|t_{\text{start}} - t_{\text{finish}}|}{h}\left(Kh^2 + \mathcal{O}(h^3)\right) \\ &= K(t_{\text{start}} - t_{\text{finish}})h + \mathcal{O}(h^2). \end{aligned}$$

This is the error after n steps ie the global error in the euler method and it is *linear*, sometimes called order 1, in step-size.

Going back to our example from earlier: $\frac{dx}{dt} = x(t)$ with $x(0) = 1$. Figure 2 shows the error in the solution after N steps ie $|x_N - x(t)|$ as a function of the stepsize h.

Further resources

Some nice online resources discussing the Euler method and errors are at

- `https://www.youtube.com/watch?v=YxAO53ND23k`
- via MIT's OCW: `https://www.youtube.com/watch?v=X5-ucBtneVM`

I'll also be posting some sample problems and solutions on Blackboard and on the course webpage.

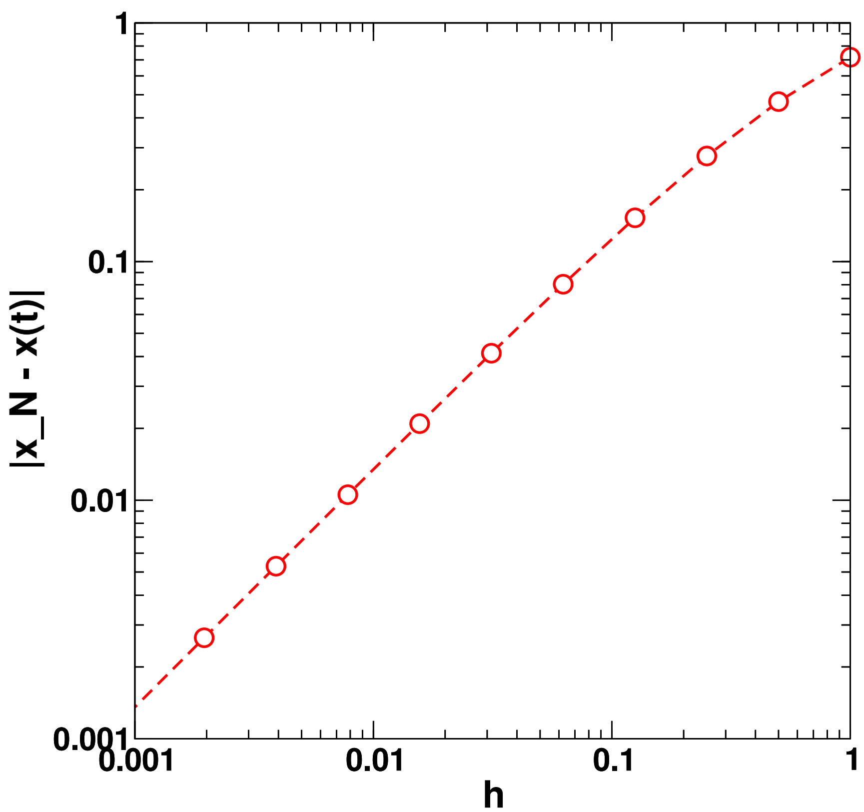

Figure 2: The error in the solution to $x' = x$ via euler method (compared to the true solution as a function of stepsize h. Note that the solution is $x(t) = e^t$ so by plotting the error on a log-log plot you see the linear behaviour of the error with h more clearly.