The microcanonical ensemble

Entropy is given by $s = k \ln \Delta\Gamma$.

We have that $\Delta\Gamma = V^N \int d^{3N}p$ and $E(p) = \frac{1}{2m}\sum_{i=1}^{N} |p_i|^2$.

Calculating Δ

Geometrically, $\Delta\Gamma = V_D^N$ where Ω_D is the volume of the D-dimensional sphere in momentum space, with radius $R = \sqrt{2mE_0}$ and thickness Δ.

Write the volume of a spherical shell in $3N$ dimensions as $V(R) = C(D)R^D$. Then the volume of a shell of thickness Δ is

$$\begin{aligned} V_\Delta &= V(R) - V(R-\Delta) \\ &= C(D)R^D\left[1 - \left(1 - \frac{\Delta}{R}\right)^D\right] \end{aligned}$$

For a gas, $D \approx 3\times 10^{23}$ so $V_\Delta \approx C(D)R^D$ and $\Delta\Gamma \approx V^D C(D)R^D$

We need $C(D)$. Get this by considering the integral

$$I(D) = \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_D e^{-(x_1^2+\ldots+x_D^2)} = \pi^{D/2}$$

In polar coordinates this is

$$\begin{aligned} I(D) &= \int_0^\infty e^{-r^2} r^{D-1} C(D) dr \\ &= \frac{1}{2}\Gamma\left(\frac{D}{2}+1\right) C(D) \end{aligned}$$

where $r^{D-1}C(D)$ is the surface area of the D-dimensional sphere. Then comparing results for $I(D)$ we have

$$C(D) = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)}$$

and

$$\Delta\Gamma = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}+1\right)} R^D V^N.$$

Using the result $\Gamma\left(\frac{D}{2}+1\right) = \left(\frac{D}{2}\right)!$ and Stirling's approximation, $N! = \sqrt{2\pi N}N^N e^{-N}$ the result follows. (Note that when taking logs of $N!$ the $ln(\sqrt{2\pi N})$ term is often ignored as its contribution is small compared to the other terms in the log.