Theorem

The Gauss-Seidel method for $Ax = b$ is convergent if A is SDD.

Proof

For Gauss Seidel

$$B = D - E$$

therefore

$$(D - E)x^{(i+1)} = b + Fx^{(i)} \tag{1}$$

and also

$$\begin{aligned} Ax &= b \\ \Rightarrow (D - E - F)x &= b \\ \Rightarrow (D - E)x &= b + Fx \end{aligned}$$

and therefore

$$(D - E)(x^{(i+1)} - x) = F(x^{(i+1)} - x)$$

We want to prove that $x^{(m)} \to x$ as $m \to \infty$ ie that $e^{(m)} \to 0$ as $m \to \infty$.

Now,

$$\begin{aligned} (D - E)e^{(m)} &= Fe^{(m-1)} \\ \Rightarrow De^{(m)} &= Ee^{(m)} + Fe^{(m-1)} \\ \Rightarrow e^{(m)} &= D^{-1}Ee^{(m)} + D^{-1}Fe^{(m-1)} \end{aligned}$$

Taking the i^{th} component of both sides we have

$$e_i^{(m)} = \frac{1}{a_{ii}} \sum_{j=1}^{i-1} (-a_{ij}) e_j^{(m)} + \frac{1}{a_{ii}} \sum_{j=i+1}^{n} (-a_{ij}) e_j^{(m-1)}$$

For $i = 1$ we have

$$e_1^{(m)} = -\frac{1}{a_{11}} \sum_{j=2}^{n} (a_{1j}) e_j^{(m-1)}$$

and hence

$$\begin{aligned} |e_1^{(m)}| &\leq \frac{1}{|a_{11}|} \sum_{j=2}^{n} |a_{1j}||e_j^{(m-1)}| \\ &\leq ||e^{(m-1)}||_\infty \frac{1}{|a_{11}|} \sum_{j=2}^{n} |a_{1j}| \\ &\leq r_1 ||e^{(m-1)}||_\infty \end{aligned}$$

where $r_1 = \frac{1}{|a_{11}|}\sum_{j=2}^{n}|a_{1j}| < 1$ by SDD hypothesis. We now define

$$r = max_{1\leq i\leq n}(r_i)$$

where $r_i = \dfrac{1}{|a_{11}|}\displaystyle\sum_{j=2}^{n}|a_{1j}|$. By SDD we have $r < 1$. We prove the theorem by induction. We assume that $|e_j^{(m)}| \leq r||e^{(m-1)}||_\infty$ for $j = 1\ldots i-1$. We have already established it for $j = 1$. We have for $i \geq 2$

$$\begin{aligned}
|e_i^{(m)}| &\leq \frac{1}{|a_{ii}|}\left(\sum_{j=1}^{i-1}|a_{ij}||e_j^{(m)}| + \sum_{j=i+1}^{n}|a_{ij}||e_j^{(m-1)}|\right)\\
&= \frac{1}{|a_{ii}|}\left(r||e^{(m-1)}||_\infty\sum_{j=1}^{i-1}|a_{ij}| + ||e^{(m-1)}||_\infty\sum_{j=i+1}^{n}|a_{ij}|\right)\\
&< ||e^{(m-1)}||_\infty\left(\frac{1}{|a_{ii}|}\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}|\right)\\
&= r_i||e^{(m-1)}||_\infty\\
&\leq r||e^{(m-1)}||_\infty
\end{aligned}$$

We conclude by induction that the result is true for all i.

Thus

$$\begin{aligned}
max_{1\leq i\leq n}|e_i^{(m)}| &\leq r||e^{(m-1)}||_\infty\\
\Rightarrow ||e^{(m)}||_\infty &\leq r||e^{(m-1)}||_\infty
\end{aligned}$$

Thus

$$||e^{(m)}||_\infty \leq r^m||e^{(0)}||_\infty$$

and so $||e^{(m)}||_\infty \to 0$ as $m \to \infty (r < 1)$ as required.