Proofs of lemmas stated in notes

For any $n \times n$ matrix A we have

Lemma: $\| \mathbf{A} \|_{\mathbf{1}} = \max\limits_{\mathbf{1 \le j \le n}} (\sum_{\mathbf{i=1}}^{\mathbf{n}} |\mathbf{a_{ij}}|)$

For any $x \in \Re^n$ we have

$$\begin{aligned}
\| Ax \|_1 &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \\
&\le \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} \| x_j| \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}| \right) |x_j| \\
&\le \overset{max}{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) \underbrace{\sum_{j=1}^{n} |x_j|}_{=\|x\|_1}
\end{aligned}$$

Thus

$$\frac{\| Ax \|_1}{\| x \|_1} \le \begin{matrix} max \\ 1 \le j \le n \end{matrix} \left(\sum_{i=1}^{n} |a_{ij}| \right) \tag{1}$$

for all $x \neq 0$.

Since the RHS does not depend on x we conclude that

$$\begin{aligned}
\| A \|_1 &= \sup_{x \neq 0} \frac{\| Ax \|_1}{\| x \|_1} &(2)\\
&\le \begin{matrix} max \\ 1 \le j \le n \end{matrix} \left(\sum_{i=1}^{n} |a_{ij}| \right) &(3)
\end{aligned}$$

Suppose now that p is such that

$$\begin{matrix} max \\ 1 \le j \le n \end{matrix} \left(\sum_{i=1}^{n} |a_{ij}| \right) = \sum_{i=1}^{n} |a_{ip}|$$

choose x^* such that $x^* = \delta_{ij}$ for all j, $1 \le j \le n$. Then $\| x \|_1 = 1$ and

$$\| Ax^* \|_1 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij} \| x_j^*| \right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |a_{ip}| \\
&= \begin{matrix} max \\ 1 \le j \le n \end{matrix} \left(\sum_{i=1}^{n} |a_{ij}| \right)
\end{aligned}
$$

and so

$$\parallel A \parallel_1 = \sup_{x \neq 0} \frac{\parallel Ax \parallel_1}{\parallel x \parallel_1} \geq \frac{\parallel Ax^* \parallel_1}{\parallel x^* \parallel_1} = \begin{matrix} max \\ 1 \le j \le n \end{matrix} \left(\sum_{i=1}^{n} |a_{ij}| \right) \tag{4}$$

The result follows from 3 and 4.

Lemma: $\parallel \mathbf{A} \parallel_\infty = \begin{smallmatrix} \mathbf{max} \\ \mathbf{1 \le i \le n} \end{smallmatrix} \left(\sum_{\mathbf{j=1}}^{\mathbf{n}} |\mathbf{a_{ij}}| \right)$

$$
\begin{aligned}
\parallel A \parallel_\infty &= \sup_{x \neq 0} \frac{\parallel Ax \parallel_\infty}{\parallel x \parallel_\infty} && (5) \\
&= \sup_{x \neq 0} \frac{\begin{smallmatrix} max \\ 1 \le i \le n \end{smallmatrix} \left| \sum_{j=1}^{n} |a_{ij}| \right|}{\begin{smallmatrix} max \\ 1 \le k \le n \end{smallmatrix} |x_k|} && (6) \\
&\leq \sup_{x \neq 0} \begin{matrix} max \\ 1 \le i \le n \end{matrix} \left[\sum_{j=1}^{n} |a_{ij}| \left(\frac{|x_j|}{\begin{smallmatrix} max \\ 1 \le k \le n \end{smallmatrix} |x_k|} \right) \right] && (7) \\
&\leq \begin{matrix} max \\ 1 \le i \le n \end{matrix} \left(\sum_{j=1}^{n} |a_{ij}| \right) && (8)
\end{aligned}
$$

Also $\parallel A \parallel_\infty \geq \dfrac{\parallel Ax^* \parallel_\infty}{\parallel x^* \parallel_\infty}$ for any given x^*.
Choose $x^* = signa_{pj}$ where p is such that

$$\begin{matrix} max \\ 1 \le i \le n \end{matrix} \left(\sum_{j=1}^{n} |a_{ij}| \right) = \sum_{j=1}^{n} |a_{pj}|$$

Clearly, $\parallel x^* \parallel_\infty = 1$ and

$$
\begin{aligned}
\frac{\parallel Ax^* \parallel_\infty}{\parallel x^* \parallel_\infty} &= \begin{matrix} max \\ 1 \le i \le n \end{matrix} \left| \sum_{j=1}^{n} a_{ij} x_j^* \right| \geq \left| \sum_{j=1}^{n} a_{pj} x_j^* \right| \\
&= \left| \sum_{j=1}^{n} a_{pj} signa_{pj} \right| = \left| \sum_{j=1}^{n} |a_{pj}| \right|
\end{aligned}
$$

$$= \sum_{j=1}^{n} a_{pj} = \underset{1 \le i \le n}{max} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$

Thus

$$\| A \|_\infty \ge \underset{1 \le i \le n}{max} \left(\sum_{j=1}^{n} |a_{ij}| \right) \tag{9}$$

The result follows from 8 and 9.

Lemma: $\mathbf{\max_{1\le i,j\le n}|a_{ij}| \le \| A \|_2 \le n \max_{1\le i,j\le n}|a_{ij}|}$

For all $x \neq 0$ we have

$$\begin{aligned}
\frac{\| Ax \|_2^2}{\| x \|_2^2} &= \frac{\sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j\right)^2}{\sum_{i=1}^n x_i^2} \le \frac{\sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}||x_j|\right)^2}{\sum_{i=1}^n x_i^2} \\
&\overset{Cauchy-Schwartz}{\le} \frac{\sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^2\right)\left(\sum_{k=1}^n |x_k|^2\right)}{\sum_{i=1}^n x_i^2} \\
&= \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 \le n^2 \underset{1 \le i,j \le n}{max} |a_{ij}|^2 \\
&= (n \underset{1 \le i,j \le n}{max} |a_{ij}|)^2
\end{aligned}$$

Hence

$$\| A \|_2 = \sup_{x\neq 0} \frac{\| Ax \|_2}{\| x \|_2} \le n \underset{1 \le i,j \le n}{max} |a_{ij}| \tag{10}$$

For any i let $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$. Let $|a_k l| = \max_{1\le i,j\le n} |a_{ij}|$. Then

$$\begin{aligned}
\| A \|_2 &= \sup_{x\neq 0_{y\neq 0}} \frac{|y^T Ax|}{\| y \|_2 \| x \|_2} \\
&\ge \frac{|e_k^T |e_l|}{\underbrace{\| e_k \|_2}_{=1} \underbrace{\| e_l \|_2}_{=1}} \\
&= |a_{kl}|
\end{aligned}$$

Thus

$$\| A \|_2 \ge \underset{1 \le i,j \le n}{max} |a_{ij}| \tag{11}$$

10 and 11 combine to give the required bounds for $\| A \|_2$.

Lemma $\| A \|_2 = \sup_{x\neq 0_{y\neq 0}} \frac{|y^*Ax|}{\|y\|_2\|x\|_2}$

$$\begin{aligned} |y^*Ax| &\leq \| y^*A \|_2 \| x \|_2 && \text{Cauchy–Schwartz} \\ &\leq \| A \|_2 \| y \|_2 \| x \|_2 \end{aligned}$$

ie. $\| A \|_2 \geq \dfrac{|y^*Ax|}{\| y \|_2 \| x \|_2}$ for all $x \neq 0, y \neq 0$.
Therefore

$$\| A \|_2 \geq \sup_{x\neq 0_{y\neq 0}} \frac{|y^*Ax|}{\| y \|_2 \| x \|_2} \tag{12}$$

Conversely, we have, on choosing $y = Ax$

$$\begin{aligned} \sup_{x\neq 0_{y\neq 0}} \frac{|y^*Ax|}{\| x \|_2 \| y \|_2} &\geq \sup_{x\neq 0_{y\neq 0}} \frac{|(Ax)^*(Ax)|}{\| x \|_2 \| Ax \|_2} \\ &= \sup_{x\neq 0} \frac{\| Ax \|_2^2}{\| x \|_2 \| Ax \|_2} \\ &= \sup_{x\neq 0} \frac{\| Ax \|_2}{\| x \|_2} \\ &= \| Ax \|_2 \end{aligned} \tag{13}$$

Equations 12 and 13 give the required result.