Proofs of lemmas stated in notes

For any $n \times n$ matrix A we have

Lemma: $\| \mathbf{A} \|_{\mathbf{1}} = \max_{\mathbf{1 \le j \le n}} \left(\sum_{\mathbf{i=1}}^{\mathbf{n}} |\mathbf{a_{ij}}| \right)$

For any $x \in \Re^n$ we have

$$\begin{aligned} \| Ax \|_1 &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \\ &\le \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij} \| x_j| \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}| \right) |x_j| \\ &\le \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) \underbrace{\sum_{j=1}^{n} |x_j|}_{=\|x\|_1} \end{aligned}$$

Thus

$$\frac{\| Ax \|_1}{\| x \|_1} \le \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) \tag{1}$$

for all $x \neq 0$.

Since the RHS does not depend on x we conclude that

$$\| A \|_1 = \sup_{x \neq 0} \frac{\| Ax \|_1}{\| x \|_1} \tag{2}$$

$$\le \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) \tag{3}$$

Suppose now that p is such that

$$\max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) = \sum_{i=1}^{n} |a_{ip}|$$

choose x^* such that $x^* = \delta_{ij}$ for all j, $1 \le j \le n$. Then $\| x \|_1 = 1$ and

$$\| Ax^* \|_1 = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij} \| x_j^*| \right)$$

$$= \sum_{i=1}^{n} |a_{ip}|$$
$$= \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

and so

$$\| A \|_1 = \sup_{x \neq 0} \frac{\| Ax \|_1}{\| x \|_1} \ge \frac{\| Ax^* \|_1}{\| x^* \|_1} = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right) \tag{4}$$

The result follows from 3 and 4.

Lemma: $\| \mathbf{A} \|_\infty = \max_{\mathbf{1 \le i \le n}} \left(\sum_{\mathbf{j=1}}^{\mathbf{n}} |\mathbf{a_{ij}}| \right)$

$$\| A \|_\infty = \sup_{x \neq 0} \frac{\| Ax \|_\infty}{\| x \|_\infty} \tag{5}$$
$$= \sup_{x \neq 0} \frac{\max_{1 \le i \le n} \left| \sum_{j=1}^{n} |a_{ij}| \right|}{\max_{1 \le k \le n} |x_k|} \tag{6}$$
$$\le \sup_{x \neq 0} \max_{1 \le i \le n} \left[\sum_{j=1}^{n} |a_{ij}| \left(\frac{|x_j|}{\max_{1 \le k \le n} |x_k|} \right) \right] \tag{7}$$
$$\le \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right) \tag{8}$$

Also $\| A \|_\infty \ge \frac{\| Ax^* \|_\infty}{\| x^* \|_\infty}$ for any given x^*.
Choose $x^* = signa_{pj}$ where p is such that

$$\max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right) = \sum_{j=1}^{n} |a_{pj}|$$

Clearly, $\| x^* \|_\infty = 1$ and

$$\frac{\| Ax^* \|_\infty}{\| x^* \|_\infty} = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j^* \right| \ge \left| \sum_{j=1}^{n} a_{pj} x_j^* \right|$$
$$= \left| \sum_{j=1}^{n} a_{pj} signa_{pj} \right| = \left| \sum_{j=1}^{n} |a_{pj}| \right|$$

$$= \sum_{j=1}^{n} a_{pj} = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$

Thus

$$\| A \|_\infty \geq \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right) \tag{9}$$

The result follows from 8 and 9.

Lemma: $\max_{1 \le i,j \le n} |a_{ij}| \le \| A \|_2 \le n \max_{1 \le i,j \le n} |a_{ij}|$

For all $x \neq 0$ we have

$$\begin{aligned}
\frac{\| Ax \|_2^2}{\| x \|_2^2} &= \frac{\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right)^2}{\sum_{i=1}^{n} x_i^2} \le \frac{\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}||x_j| \right)^2}{\sum_{i=1}^{n} x_i^2} \\
&\overset{Cauchy-Schwartz}{\le} \frac{\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right) \left(\sum_{k=1}^{n} |x_k|^2 \right)}{\sum_{i=1}^{n} x_i^2} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \le n^2 \max_{1 \le i,j \le n} |a_{ij}|^2 \\
&= (n \max_{1 \le i,j \le n} |a_{ij}|)^2
\end{aligned}$$

Hence

$$\| A \|_2 = \sup_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2} \le n \max_{1 \le i,j \le n} |a_{ij}| \tag{10}$$

For any i let $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$. Let $|a_k l| = \max_{1 \le i,j \le n} |a_{ij}|$. Then

$$\begin{aligned}
\| A \|_2 &= \sup_{x \neq 0 y \neq 0} \frac{|y^T A x|}{\| y \|_2 \| x \|_2} \\
&\geq \frac{|e_k^T| e_l|}{\underbrace{\| e_k \|_2}_{=1} \underbrace{\| e_l \|_2}_{=1}} \\
&= |a_{kl}|
\end{aligned}$$

Thus

$$\| A \|_2 \geq \max_{1 \le i,j \le n} |a_{ij}| \tag{11}$$

10 and 11 combine to give the required bounds for $\| A \|_2$.

Lemma $\| A \|_2 = \sup_{x \neq 0_{y \neq 0}} \frac{|y^* A x|}{\|y\|_2 \|x\|_2}$

$$\begin{aligned} |y^* A x| &\leq \| y^* A \|_2 \| x \|_2 \qquad \text{Cauchy–Schwartz} \\ &\leq \| A \|_2 \| y \|_2 \| x \|_2 \end{aligned}$$

ie. $\| A \|_2 \geq \dfrac{|y^* A x|}{\| y \|_2 \| x \|_2}$ for all $x \neq 0, y \neq 0$.
Therefore

$$\| A \|_2 \geq \sup_{x \neq 0_{y \neq 0}} \frac{|y^* A x|}{\| y \|_2 \| x \|_2} \tag{12}$$

Conversely, we have, on choosing $y = Ax$

$$\begin{aligned} \sup_{x \neq 0_{y \neq 0}} \frac{|y^* A x|}{\| x \|_2 \| y \|_2} &\geq \sup_{x \neq 0_{y \neq 0}} \frac{|(Ax)^* (Ax)|}{\| x \|_2 \| Ax \|_2} \\ &= \sup_{x \neq 0} \frac{\| Ax \|_2^2}{\| x \|_2 \| Ax \|_2} \\ &= \sup_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2} \\ &= \| Ax \|_2 \end{aligned} \tag{13}$$

Equations 12 and 13 give the required result.