UNIVERSITY OF DUBLIN

2134

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics **Hilary Term 2001**
Foundation Scholarship

COURSE 261

Monday, 26 March EXAM HALL 09:30 – 12.30

Dr. S. Ryan

Answer 5 Questions

1. Given the 3×3 matrix A and the vector, b

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}$$

Use Gauss-Jordan elimination to solve the system $Ax = b$ for the vector x and to find the matrix inverse, A^{-1}.

Determine the condition number, $c_\infty(A)$ and comment on whether or not the matrix A is well-conditioned, with respect to the norm, $\|\cdot\|_\infty$.

2. Answer 4 of the 6 parts, *(a)* to *(f)*:
Consider the linear system $Ax = b$, where A is a non-singular matrix of order n.

   (a) Describe the Jacobi and Gauss-Seidel techniques to solve a matrix equation $Ax = b$ for x.

   (b) Prove the Jacobi method is convergent if the matrix A is Strictly Diagonally Dominant (SDD).

   (c) Determine if the matrix, A, in the question 1 is SDD.

   (d) For either the Jacobi or the Gauss-Seidel Method, sketch, in pseudo-code, one step of the numerical algorithm to implement the scheme.

   (e) Describe briefly Krylov methods and, in particular, how they differ from the Jacobi and Gauss-Seidel methods.

(f) Applying Krylov subspace techniques to solve $Ax = b$ derive the Arnoldi algorithm for partial reduction to upper Hessenberg form. You can assume that the orginal Krylov matrix K has been replaced by the orthogonal matrix Q such that for all k, the leading k columns of K and Q span the Krylov space and $Q^T AQ = H$, an upper Hessenberg matrix.

3. Prove that for $n+1$ arbitrary points, (x_i, f_i), $i = 0, 1, \ldots, n$ and $x_i \neq x_k$ there exists a unique polynomial $P \in \Pi_n$, the space of all polynomials of degree not greater than n, with $P(x_i) = f_i$.

   Given the data

x_i	0	2	4
f_i	1	3	5

   Use Neville's interpolating algorithm to determine $P(1)$.

   Use Newton's divided difference algorithm to determine the form of the interpolating polynomial and also $P(3)$.

4. Given the rational function expression

$$\Phi^{\mu,\nu}(x) = \frac{P^{\mu,\nu}(x)}{Q^{\mu,\nu}(x)} = \frac{a_0 + a_1 x + \ldots + a_\mu x^\mu}{b_0 + b_1 x + \ldots + b_\nu x^\nu},$$

   with $\Phi^{\mu,\nu}(x_i) = f_i$ for $i = 0, 1 \ldots \mu + \nu$.
   If the homogeneous system of linear equations $P^{\mu,\nu}(x_i) - f_i Q^{\mu,\nu}(x_i) = 0$ is denoted by $S^{\mu,\nu}$ prove that $S^{\mu,\nu}$ always has non-trivial solutions and that all non-trivial solutions define rational expressions.

   Prove that if Φ_1 and Φ_2 are non-trivial solutions of $S^{\mu,\nu}$ then they are equivalent (ie. $\Phi_1 \sim \Phi_2$).

   Complete the tableau below by calculating the inverse differences from the given support points (x_i, f_i).

i	x_i	f_i	$\psi(x_0, x_i)$	$\psi(x_0, x_1, x_i)$	$\psi(x_0, x_1, x_2, x_i)$
0	0	0			
1	1	-1			
2	2	$-2/3$			
3	3	9			

   Using these inverse differences and the notation of Thiele's continued fractions write down the expression for the rational interpolating polynomial, $\Phi^{2,1}(x)$.

5. Show that Simpson's Rule for numerical integration is an order 4 method.

   Determine the Peano kernel, $K(t)$ for Simpson's rule and $n = 3$ in $[-1, +1]$. Does it change sign in the interval of integration?

6. For numerical integration methods, write down the asymptotic expansion in h (step-length) of the result, $T(h)$.

   Derive an expression for the error, $|T_{ik} - \tau_0|$, in such methods. As part of the derivation you should prove that

   (i) The Lagrange coefficients, $c_{kj}^{(i)}$ in $T_{ik} = \sum_{j=i-k}^{i} c_{kj}^{(i)} T(h_j)$ ie when $z = h^\gamma = 0$ are given by

   $$\sum_{j=i-k}^{i} c_{kj}^{(i)} z_j^\tau = \begin{cases} 1 & \text{if } \tau = 0 \\ 0 & \text{if } \tau = 1, 2, \ldots, k \\ (-1)^k z_{i-k} z_{i-k+1} \ldots z_i & \text{if } \tau = k+1 \end{cases}$$

   (ii) For sequences with $\dfrac{h_{j+1}}{h_j} \leq b < 1;; \forall j$ and in particular with $h_j = h_0 b^j, 0 < b < 1$ that

   $$\sum_{j=i-k}^{i} c_{kj}^{(i)} z_j^{k+1} \leq C_k z_{i-k} z_{i-k+1} \ldots z_i.$$

   Show that when the following inequality

   $$0 \leq \frac{|T_{i+1,k} - \tau_0|}{|T_{ik} - \tau_0|} \sim \frac{h_{i+1}^\gamma}{h_{i-1}^\gamma} \leq b^{\gamma(k+1)} < \frac{1}{2}$$

   holds, there is a function U_{ik} such that the desired value τ_0 is bracketed by U_{ik} and T_{ik}.