UNIVERSITY OF DUBLIN

TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Mathematics
JF Theoretical Physics
JF Two Subject Mod

Michaelmas Term 2006

Course 111

Wednesday, December 13 Regent House 09.30 — 11.30

Dr. S. Ryan

ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

Note: This is a sample paper for the Christmas exam so you can see the format and style of questions you might be asked.

The questions here are drawn from your tutorial questions and class notes. You shouldn't interpret this paper as an indication of what material may, or may not, appear on the exam.

1. **Equivalence Relation:** Given a non-empty set A and a nonempty subset $R \subset A \times A$. If $(a, b) \in R$ then $a \sim b$ defines an equivalence relation if (i) $a \in A$ then $a \sim a$ (reflexivity; (ii) $a \sim b$ then $b \sim a$ (symmetry); (iii) $a \sim b, b \sim c$ then $a \sim c$ (transitivity).

   Given $\sim$ an equivalence relation on a set A and $a \in A$. An **equivalence class** containing a is defined by $cl(a) = \{x \in A | a \sim x\}$.

   Since $a - a = 0$ an even integer, then $a \sim a$ and the relation is reflexive. If $a \sim b$ then $(a - b)$ is an even integer and then $b - a = -(a - b)$ is also even and so $b \sim a$ and the relation is symmetric. If $a \sim b$ and $b \sim c$ then both $(a - b)$ and $(b - c)$ are even and so $(a - c) = (a - b) + (b - c)$ is also even and the relation is transitive.

   The equivalence class of a is all integers of the form $a + 2m$ for $m = 0, \pm 1, \pm 2, \ldots$. Then there are two distinct equivalence classes, $cl(0)$ and $cl(1)$.

2. A map from a set S to a set T is a subset M of $S \times T$ such that for every $s \in S$ there is a unique $t \in T$ such that the ordered pair $(s,t) \in M$. A map that is one-to-one and onto is a bijective map.

   The map, $\sigma : R \to R$ defined by $\sigma(x) = 2x + 1$ is a bijection from the set of real numbers to itself.

   The map $\sigma : R \to R$ defined by $\sigma(x) = x^3 - x$ is not one to one since $-1, 0, 1 \in R$ and all map to 0. Therefore the map is not a bijection.

   Consider two maps from a set S to a set T and from the set T to a set U, $\sigma : S \to T$ and $\tau : T \to U$. Prove that $\sigma \circ \tau$ is one-to-one if each of σ and τ is one-to-one.

   Proof: Suppose $s_1, s_2 \in S$ and $s_1 \neq s_2$. Then since σ is one to one this means that $s_1\sigma \neq s_2\sigma$. Since τ is one to one and $s_1\sigma, s_2\sigma$ are distinct elements of T then $(s_1\sigma)\tau \neq (s_2\sigma)\tau$.

   Therefore, $s_1(\sigma \circ \tau) = (s_1\sigma)\tau) \neq (s_2\sigma)\tau = s_2(\sigma \circ \tau)$.

3. A nonempty set, G forms a group if there is a defined binary operation, $\phi : G \times G \to G$, written (.) such that

   - if $a, b \in G$ then $a.b \in G$
   - $a, b, c \in G$ then $a.(b.c) = (a.b).c$
   - there is an $e \in G$ such that $a.e = e.a = a$ for all $a \in G$
   - for each $a \in G$ there is an $a^{-1} \in G$ such that $a.a^{-1} = a^{-1}.a = e$.

$o(e) = 1$ since $e^1 = e$

$o(a) = 4$ since $a.a.a.a = a^4 = e$

$o(a^2) = 2$ since $a^2.a^2 = a^4 = e$

$o(a^3) = 4$ since $a^3.a^3.a^3.a^3 = (a^4)^3 = e$

The only proper subgroup of G is $\{e, a^2\}$. ie it contains the group identity element, e, it is closed under the group operation, $a^2.e = a^2, a^2.a^2 = a^4 = e, e.e = e$ and each element has an inverse since $e.e = e$ so $e^{-1} = e$ and $a^2.a^2 = e$ so $(a^2)^{-1} = a^2$.
In addition, $\{e\}, \{e, a, a^2, a^3\}$ are also (improper) subgroups.

List the distinct left cosets of the subgroup $H = \{e, a^2\}$ in G.

Given $H = \{e, a^2\}$ the left cosets are $\{e, a^2\}, \{a, a^3\}, \{a^2, a^4 = e\}, \{a^3, a^5 = a\}$ so the two distinct left cosets are

$\{e, a^2\}$ and $\{a, a^3\}$.

4. A nonempty subset H of a group G is a subgroup iff

- $a, b \in H \Rightarrow a.b \in H$
- $a \in H \Rightarrow a^{-1} \in H$.

Prove that for H and K both subgroups of a group G. The product, HK is itself a subgroup of G iff $HK = KH$.

Proof:

Begin by supposing that $HK = KH$ - then we must prove that HK is a subgroup.

$HK = KH$ means that if $h \in H$ and $k \in K$ then $hk = k_1h_1$ for some $k_1 \in K$ and $h_1 \in H$. Note this doesn't require that $k_1 = k$ or that $h_1 = h$. To prove that HK a subgroup we have to show it is closed and that every element in HK has an inverse in HK.

- **closure:** suppose $x = hk \in HK$ and $y = h'k' \in HK$. Then $xy = hkh'k'$. But now since $kh' \in KH = HK$ we have $kh' = h_2k_2$ with $h_2 \in H$ and $k_2 \in K$. Hence $xy = h(h_2k_2)k' = (hh_2)(k_2k') \in HK$ and so HK is closed.
- **inverses:** $x^{-1} = (hk)^{-1} = k^{-1}h^{-1} \in KH = HK$. So, $x^{-1} \in HK$.

Then, HK a subgroup.

Now suppose HK is a subgroup - then we must show $HK = KH$.

If HK a subgroup of G then for any $h \in H$ and $k \in K$, $h^{-1}k^{-1} \in HK$ and so $kh = (h^{-1}k^{-1})^{-1} \in HK$. Thus $KH \subset HK$.

Now if x is any element of HK then $x^{-1} = hk \in HK$ and so $x = (x^{-1})^{-1} = (hk)^{-1} = k^{-1}h^{-1} \in KH$ and so $HK \subset KH$.

Thus, $HK = KH$.

Verify that $H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \mid ad \neq 0 \right\}$ is a subgroup of the group of 2×2 matrices expressed as, $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0 \right\}$ under matrix multiplication.

Solution Must show:

- $a, b \in H \Rightarrow a.b \in H$
- $a \in H \Rightarrow a^{-1} \in H$.

Consider $h = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $k = \begin{pmatrix} x & y \\ 0 & w \end{pmatrix}$ both in H. Then

$$h.k = \begin{pmatrix} ax & ay + bw \\ 0 & dw \end{pmatrix} \tag{1}$$

an element of H.

Also

$$h^{-1} = \frac{1}{ad} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$$

and element of H.

Thus $H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \mid ad \neq 0 \right\}$ is a subgroup as required.