UNIVERSITY OF DUBLIN

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TRINITY COLLEGE

Faculty of Science

school of mathematics

JF Mathematics **Hilary Term 2007**
JF Theoretical Physics
JF Two Subject Mod

Course 111

Monday, March 12 EXAM HALL 09.30 — 11.30

Dr. S. Ryan

ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. Given the set of vectors $W = \{v_1, \ldots, v_n\}$ which span the vector space, V. Prove that

   (i) W is a subspace of V.

   (ii) W is the smallest subspace of V containing all vectors, $\{v_1, \ldots, v_n\}$.

   Consider the vectors $v_1 = (1,1,0), v_2 = (5,1,-3), v_3 = (2,7,4)$ in $\Re$ (where $\Re$ is the set of all real numbers). Determine if these vectors are linearly independent and if they span the vector space, $\Re^3$.

2. Consider the system of linear equations

$$\begin{aligned} -2x + y + z &= -5, \\ x + z &= 5, \\ x - 3y - 2z &= 8. \end{aligned}$$

   Write this system as a matrix equation and use Gauss-Jordan elimination to solve for x, y and z.

   Define the row space of an $m \times n$ matrix A with entries $a_{ij} \in \Re$. Consider a matrix B which can be obtained from A by elementary row operations. Prove that the row space of B is identical to that of A.

   Consider the coefficient matrix of the system of linear equations given above and call it A. Determine a basis for the row space of A.

   What is the rank of A?

3. Consider an $n \times n$ matrix A which can be reduced to row-echelon form without interchanging rows. Prove that A can be written as $A = LU$ where L is a lower triangular, $n \times n$ matrix and U is an upper triangular, $n \times n$ matrix.

   Use LU decomposition to determine the inverse of the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix}.$$

   Sketch briefly how LU decompostion can be used to solve a system of linear equations.

4. Suppose that A is an $n \times n$ matrix. Prove that A is invertible if and only if $\mathtt{det}(A) \neq 0$, where $\mathtt{det}(A)$ is the determinant of the matrix A.

   Note: you may assume the result $\mathtt{det}(AB) = \mathtt{det}(A)\mathtt{det}(B)$, for A and B both $n \times n$ matrices, if this is needed in your proof.

   Consider the matrix
$$A = \begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix}.$$

   Show that it has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$ with corresponding eigenvectors $v_1 = (-2, 1)$ and $v_2 = (-8, 1)$, respectively.

   Prove that an $n \times n$ matrix A with n linearly independent eigenvectors $v_1, \ldots, v_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ can be written as
$$A = S\Lambda S^{-1},$$

   where S is the $n \times n$ matrix whose columns are the eigenvectors, $v_1, \ldots, v_n$ and Λ is the $n \times n$ diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$.

   Use this result and the eigenvalues and eigenvectors already determined for A above to determine A^6.