UNIVERSITY OF DUBLIN

XMA1111

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JF Mathematics **Hilary Term 2007**
JF Theoretical Physics
JF Two Subject Mod

COURSE 111

Monday, March 12 EXAM HALL 09.30 — 11.30

Dr. S. Ryan

ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. Given the set of vectors $W = \{v_1, \dots, v_n\}$ which span the vector space, V. Prove that

   (i) W is a subspace of V.

   (ii) W is the smallest subspace of V containing all vectors, $\{v_1, \dots, v_n\}$.

   (i)

   W a subspace of V ie W is a subspace if $\alpha w_1 + \beta w_2 \in W$. So must show W is closed on addition and multiplication.

   Let $r, w \in W$ then

$$\begin{aligned} r &= c_1v_1 + c_2v_2 + \dots + c_nv_n \\ w &= k_1v_1 + k_2v_2 + \dots + k_nv_n \\ \Rightarrow r + w &= (c_1 + k_1)v_1 + \dots + (c_n + k_n)v_n \text{ ie } \in E \end{aligned}$$

   Similarly

$$kr = (kc_1)v_1 + (kc_2)v_2 + \dots + (kc_n)v_n$$

   so kr a linear combination of v_i and so $\in W$.

   Therefore W a subspace.

   (ii)

   Consider W' a vector space with $v_1 \dots v_n$ and consider $u \in W$. If $u \in W'$ then W' contains a copy of W.

$$u \in W \Rightarrow u = c_1v_1 + c_2v_2 + \dots + c_nv_n$$

   each of the $c_iv_i \in W'$ bu (i) and so u is the sum of vectors in W' and so $u \in W'$. Then W' contains every vector from W so contains W.

   Consider the vectors $v_1 = (1, 1, 0), v_2 = (5, 1, -3), v_3 = (2, 7, 4)$ in $\Re$ (where $\Re$ is the set of all real numbers). Determine if these vectors are linearly independent and if they span the vector space, $\Re^3$.

   An easy way to show this is to consider the expressions $c_1(1, 1, 0) + c_2(5, 1, -3) + c_3(2, 7, 4) = 0$ and $c_1(1, 1, 0) + c_2(5, 1, -3) + c_3(2, 7, 4) = (u_1, u_2, u_3$. Solving for the c_i in the former and if the $c_i = 0$ implies linear independence. If there are c_i such that a vector u_i can be expressed as in the latter equation proves they span.

   Writing the expressions as matrix equations.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{1}$$

It is easy to easy show that $c_1 = c_2 = c_3 = 0$ is a unique solution.

Similarly

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{2}$$

The determinant of the 3x3 matrix is nonzero therefore it is invertible and so there are c_i such that $c_1(1,1,0) + c_2(5,1,-3) + c_3(2,7,4) = (u_1, u_2, u_3)$.

2. Consider the system of linear equations

$$\begin{aligned} -2x + y + z &= -5, \\ x + z &= 5, \\ x - 3y - 2z &= 8. \end{aligned}$$

Write this system as a matrix equation and use Gauss-Jordan elimination to solve for x, y and z.

I'll just write the row operations and the solutions. $R_3 - R_2$, $R_2 - \frac{1}{2}R_1$, $R_2 - R_1$, $R_1 + R_2$, $2R_2$, $R_3 + 3R_2$, $R_2 - \frac{1}{2}R_3$, $R_1 - \frac{1}{6}R_3$, $\frac{1}{6}R_3$. Giving the solution $x = 2, y = -4, z = 3$.

Define the row space of an $m \times n$ matrix A with entries $a_{ij} \in \Re$.

The row space, $R(A)$, is the subspace of R^n spanned by the rows of A.

Consider a matrix B which can be obtained from A by elementary row operations. Prove that the row space of B is identical to that of A.

The rows of B are linear combinations of rows of A so any linear combination of rows of B is a linear combination of rows of A. $\Rightarrow R(B) \subset R(A)$. Also the rows of A are linear combinations of rows of B (since the e.r.o's are invertible). $\Rightarrow R(A) \subset R(B)$.

$\Rightarrow R(A) = R(B)$.

Consider the coefficient matrix of the system of linear equations given above and call it A. Determine a basis for the row space of A. One solution is the write A in REF

$$A = \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Then (argued in notes) $v_1 = (1, -1/2, -1/2), v_2 = (0, 1, 3), v_3 = (0, 0, 1)$ is a basis of $R(A)$.

What is the rank of A?

3.

3. Consider an $n \times n$ matrix A which can be reduced to row-echelon form without interchanging rows. Prove that A can be written as $A = LU$ where L is a lower triangular, $n \times n$ matrix and U is an upper triangular, $n \times n$ matrix.

Consider A reduced to upper triangular form by ero's. Each ero can be represented as an elementary matrix, E_i. Then

$$\begin{aligned} E_1 E_2 \dots E_k A &= U \\ \Rightarrow A &= E_k^{-1} E_{k-1}^{-1} \dots E_1^{-1} U \end{aligned}$$

and each E_i is lower triangular, so are the E_i^{-1} and the product of lower triangular matrices, $E_k^{-1} E_{k-1}^{-1}$ is also lower triangular. Writing this product as L then

$$A = LU$$

as required.

Use LU decomposition to determine the inverse of the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix}.$$

To calculate the inverse via LU decomposition

$$\begin{aligned} AA^{-1} &= I \\ LUA^{-1} &= I \\ A^{-1} &= U^{-1} L^{-1} \end{aligned}$$

Then,

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 4 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} x_1 & 0 & 0 \\ x_4 & x_2 & 0 \\ x_5 & x_6 & x_3 \end{pmatrix} \begin{pmatrix} 1 & y_1 & y_2 \\ 0 & 1 & y_3 \\ 0 & 0 & 1 \end{pmatrix}$$

Solving, eg using Crout's algorithm gives: $x_1 = 2, x_2 = -5, x_3 = 43/10, x_4 = 4, x_5 = -5, x_6 = -1/2, y_1 = 3/2, y_2 = 1/2, y_3 = -2/5$. Now we know the components of U and L we can use ero's to determine their inverses. Here I just list the ero's and resulting the inverse matrices.

For U^{-1}: $R_1 - 3/2R_2, R_1 - 11/10R_3, R_2 + 2/5R_3$ gives

$$U^{-1} = \begin{pmatrix} 1 & -3/2 & -11/10 \\ 0 & 1 & 2/5 \\ 0 & 0 & 1 \end{pmatrix}$$

For L^{-1}: $1/2R_1, R_2 - 4R_1, R_3 - 3R_1, -1/5R_2, R_3 + 1/2R_2, 10/43R_3$ gives

$$L^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 2/5 & -1/5 & 0 \\ -13/43 & -1/43 & 10/43 \end{pmatrix}$$

Then,

$$A^{-1} = U^{-1}L^{-1} = 1/43 \begin{pmatrix} 10 & 14 & -11 \\ 12 & -9 & 4 \\ -13 & -1 & 10 \end{pmatrix}$$

Sketch briefly how LU decompostion can be used to solve a system of linear equations.

Want to solve $Ax = b$ which can be written $LUx = b$. Write $Ux = y$ then solve $Ly = b$ for y. Knowing y solve $Ux = y$ for x as required.

4. Suppose that A is an $n \times n$ matrix. Prove that A is invertible if and only if $\texttt{det}(A) \neq 0$, where $\texttt{det}(A)$ is the determinant of the matrix A.

   Note: you may assume the result $\texttt{det}(AB) = \texttt{det}(A)\texttt{det}(B)$, for A and B both $n \times n$ matrices, if this is needed in your proof.

   Suppose A is invertible then

$$\begin{aligned} AA^{-1} &= I \\ det(AA^{-1}) &= 1 \\ det(A)det(A^{-1}) = 1 \\ det(A^{-1}) = 1/det(A) \end{aligned}$$

   and so $det(A) \neq 0$.

   Now suppose $det(A) \neq 0$ then

$$A_{RREF} = E_k \ldots E_1 A$$

   ie A can be reduced to RREF by elementary matrices (ero's). From this equation we can write $det(A_{RREF}) = det(E_k) \ldots det(E_1)det(A)$, using the fact that the E_i are invertible means $det(E_i)$ exists so $det(A_{RREF}) \neq 0$ and therefore A_{RREF} has no zero rows (a property of the determinant). This means that $A_{RREF} = I$ and so A is row equivalent to the identity matrix. This implies A is invertible.

   Consider the matrix

$$A = \begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix}.$$

   Show that it has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$ with corresponding eigenvectors $v_1 = (-2, 1)$ and $v_2 = (-8, 1)$, respectively.

   Have $det(A - \lambda I) = (6 - \lambda)(-4 - \lambda) + 16 = (\lambda - 4)(\lambda + 2) = 0$. So, $\lambda_1 = -2$ and $\lambda_2 = 4$. To find the eigenvectors, solve $(A - \lambda I)x = 0$ for the eigenvector x associated to each eigenvalue. You get the results above.

   Prove that an $n \times n$ matrix A with n linearly independent eigenvectors $v_1, \ldots, v_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ can be written as

$$A = S\Lambda S^{-1},$$

   where S is the $n \times n$ matrix whose columns are the eigenvectors, $v_1, \ldots, v_n$ and Λ is the $n \times n$ diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$.

   Need to show that $AS = S\Lambda$

$$\begin{aligned} AS &= A[v_1, v_2, \ldots, v_n] \\ &= [Av_1, Av_2, \ldots, Av_n] \\ &= [\lambda v_1, \lambda v_2, \ldots, \lambda v_n] \\ &= [v_1, v_2, \ldots, v_n] \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots \\ \vdots & \ldots & & \lambda_n \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= S\Lambda \\ \Rightarrow A &= S\Lambda S^{-1} \end{aligned}$$

as required.

Use this result and the eigenvalues and eigenvectors already determined for A above to determine A^6.

$$\begin{aligned} A^6 &= S\Lambda^6 S^{-1} \\ &= \begin{pmatrix} -2 & 8 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2^6 & 0 \\ 0 & 4^6 \end{pmatrix} \begin{pmatrix} 1/6 & 8/6 \\ -1/6 & -2/6 \end{pmatrix} \\ &= \begin{pmatrix} 5440 & 10752 \\ -672 & -1280 \end{pmatrix} \end{aligned}$$

Note I worked out S^{-1} easily using Cramer's rule.