MA3429 Differnetial Geometry 1

Ruairi Short

September 27th, 2011

Will be essentially Differential Geometry for General Relativity. DG, in particular, tensor calculus is the mathematical framework of GR.

Contents

1 Manifolds

1.1 Topological Space

Let p$\in \mathbb{R}^n$. A nbh[1] of p is any set $V \subset \mathbb{R}^n$ such that V contains an open solid sphere of centre p.
Properties of nbh's:

1. p belongs to any nbh of p
2. If V is a nbh of p and $V \subset U$, then U is a nbh of p.

[1]neighbourhood

3. If U,V are nbh's of p, then $U \cap V$ is a nbh of p.
4. If U is a nbh of p there is a nbh V of p s.t. $V \subset U$, and V is a nbh of each of its points.

Definition 1. *A topological space is a set of points M along with an assignment to each p∈M of collections od subsets called nbh's, satisfying properties 1-4.*

1.2 Charts

Let M be a topological space, p$/in$M be some point in this space, U be an open nbh of p. A chart on U is a one-to-one(injective) map:

$$\phi : U \to \phi(U) \subset \mathbb{R}^n$$

The $\phi(p) \in \mathbb{R}^n$ constitutes a local co-ordinate system defined in an open nhb U, we usually write $\phi(p) = \{x^\mu\} = \{x^1(p), x^2(p), \ldots, x^n(p)\}$
NB: The choice of chart is arbitrary
→Einstein Equivalence Principle.

1.3 Meshing Condition: Coordinate Transformations

Suppose we have two charts ϕ_1, ϕ_2 on U⊂M. Since these charts ae in jective they are invertible e.g.

$$\phi_1^{-1} : \phi_1(U) \subset \mathbb{R}^n \to U$$

We may define

$$\phi_2 \circ \phi_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n : \phi_1(U) \to \phi_2(U)$$

We require these maps to be smooth (C^∞) where they are defined. For p∈U, the map $\phi_2 \circ \phi_1^{-1}(p)$ defines a co-ord transformation from the co-ords

$$\phi_1(p) = \{x^1(p), x^2(p), \ldots, x^n(p)\}$$

to the co-ords

$$\phi_2(p) = \{X^1(p), X^2(p), \ldots, X^n(p)\}$$

1.3.1 Example

Let M∈ $\mathbb{R}^2$, let ϕ_1 be a map p to Cartesian coords (x,y) and ϕ_2 map p to a different set of cartesian coords (X,Y) obtained from the first by a rotation through the angle α.

$$\phi_2 \circ \phi_1^{-1} : (x, y) \to (X = x \cos\alpha + y \sin\alpha, Y = -x \sin\alpha + y \cos\alpha)$$

We can define a derivative matrix

$$D(\phi_2 \circ \phi_1^{-1}) = \begin{pmatrix} \frac{\partial X}{\partial x} & \frac{\partial Y}{\partial y} \\ \frac{\partial Y}{\partial x} & \frac{\partial X}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

Where Jacobian J $\equiv$ det(D)=1

Reminder J $/neq$ 0 implies invertible transformation. J non-singular[2] implies ϕ_1, ϕ_2 are C^{infty} related Introduce another chart ϕ_3, which maps p to polar co-ords (r, θ).

$$\phi_3 \circ \phi_1^{-1} : (x, y) \to (r = \sqrt{x^2 + y^2}, \theta = \tan \frac{y}{x})$$

J=det(D)=$\frac{1}{r}$ ϕ_1, ϕ_1 are C^∞ related except at $r = 0$. To cover all of $/mathdsR^2$, we would need at least two sets of polar co-ords with different origins.

1.3.2 Example: Stereographic Projection

$$x^2 + y^2 + z^2 = 1$$

describes a 2-sphere embedded in $\mathbb{R}^3$.

$$\phi_1(x, y, z) \equiv (w^1, w^2) = \left(\frac{2x}{1-z}, \frac{2y}{1-z} \right)$$

maps $U : S^2 - \{0, 0, 1\} \to \phi_1(U)(\mathbb{R}^n)$ [3]

1.4 Definition of a manifold

Informally a monifold is a set of points M that locally looks like a subset of $\mathbb{R}^n$. The simplest example of a curved manifold is S^2. A set of C^∞-relatd charts s.t. every pointp$\in$M lies in the domain of at least one chart is a C^∞-atlas for M. The union of all such atlases is the C^∞ maximal atlas.

Definition 2. *We define a C^∞ n-dimensional manifold by a set M along with a maximal atlas.*

2 Tangent Vectors and Tangent Spaces

In our familiar treatmane tof vectors in $\mathbb{R}^n$ they represent "directed magnitudes". This idea is no longer useful in DG, rather to each point p$\in$M, we have a set of all possible vectors at p known as the tangent space, $T_p(M)$.
Prefer to describe geometry of M from intrinsic properties alone, we won't rely on embedding in a hgher dimensional space.

[2] has no singularities

[3] defined for every point on 2-sphere except north pole. do the same with $z = 1$ plane except from suoth pole to get entire map. (on first assignment)

2.1 Smooth Function

Let M be a manifold, f be a real function

$$f : M \to \mathbb{R}$$

How do we define the "smoothness" of f?
We introduce a chart ϕ. We define F s.t.

$$F : \mathbb{R}^n \to \mathbb{R},\ F = f \circ \phi^{-1}$$

We say that f is smooth iff F is smooth in the usual sense.

Theorem 1. *The smoothness of f is chart independent.*

Proof 1. *Let ϕ_1, ϕ_2 be two meshing charts.*

$$F_i = f \circ \phi_1^{-1}\ i = 1, 2$$

$$F_1 = f \circ \phi_1^{-1}$$

$$= f \circ \phi_2^{-1} \circ \phi_2 \circ \phi_1^{-1}$$

$$F_1 = F_2 \circ \phi_2 \circ \phi_1^{-1}$$

ϕ_1, ϕ_2 are meshing charts and thus smooth. Smoothness properties of F_1 are thus the same as F_2. ■

Definition of smooth functions may be generalised toa function mapping a manifold M to another manifold N.

$$f : M \to N$$

Let ϕ_1 be a chart in M (dim n_1)
Let ϕ_2 be a chath in N (din n_2)
Define $F = \phi_2 \circ f \circ \phi_1^{-1}$, f smooth iff F smooth. Easy to prove that this is chart independent.
NB [4]

$$\frac{\partial f}{\partial x^\mu} = \frac{\partial F}{\partial x^\mu} = \frac{\partial(\phi_2 \circ f \circ \phi_1^{-1})}{\partial x^\mu}$$

2.2 Smooth Curves

Let $I = (a, b)$ be an interval of $\mathbb{R}$. We define a curve in M as a map

$$\gamma : \mathbb{R} \to M,\ \gamma(s) \to p$$

The curve γ is smooth if its image

$$\phi \circ \gamma : T \to \mathbb{R}^n,\ \phi \circ \gamma(s) = \{x^1(s), x^2(s), \ldots, x^n(s)\}$$

is smooth

[4] the first part of the following is not really defined on manifolds but is kinda ok to write becasue they look locally like $\mathbb{R}^n$

2.3 Tangent Space as a Space of directional Derivatives

We wish to construct the tangent space at p∈M (i.e. $T_p(M)$) using inly the intrinsic properties to M.
We combine the concept of smooth functions f, and smooth curves, γ and define

$$\mathcal{F} : I \to \mathbb{R},\ s \to \mathbb{F}(s) = f \circ \gamma(s) \equiv f(\gamma(s))$$

i.e.$\mathcal{F}$ evaluates f along the curve γ
The rate at which $mathcalF$ changes, $\frac{d\mathcal{F}}{ds}$ gives the rate of change of f following the curve. The tangent vector to the curve γ at p where (w.l.o.g.[5], take $s = 0$ at p) is the real map from the set of real functions to $\mathbb{R}$, defined by

$$\dot{\gamma}_p f :\to \dot{\gamma}_p f \equiv \dot{\gamma}_p(f) = \left[\frac{d}{ds} f \circ \gamma\right]_{s=0}$$

$$\equiv \left(\frac{d\mathcal{F}}{ds}\right)_{s=0}$$

Claim Let ϕ be a chart s.t. $\phi : \phi(p) \to x^\mu(p)$
Then

$$\dot{\mathcal{F}}(0) \equiv \left[\frac{d}{ds}(f \circ \gamma)\right]_{s=0}$$

$$= \sum_{\mu=1}^{n} \left(\frac{\partial \mathcal{F}}{\partial x^\mu}\right)_{\phi(p)} \left[\frac{d}{ds} x^\mu(\gamma(s))\right]$$

where $F = f \circ \phi^{-1}$
Proof

$$\mathcal{F}(s) = f \circ \gamma = f \circ \phi^{-1} \circ \phi \circ \gamma$$

$$= F \circ \phi\gamma(s)$$

The function $\gamma \circ \phi$ maps s to the coords of $\gamma(s)$
Identify $\mathcal{F} = F\{x^1(\gamma(s)), x^2(\gamma(s)), \ldots, x^n(\gamma(s))\}$

$$\dot{\gamma}_p f = \left[\frac{d\mathcal{F}}{ds}\right]_{s=0} = \left[\frac{dF(x^1(\gamma(s)), \ldots, x^n(\gamma(s))}{ds}\right]_{s=0}$$

Chain Rule:

$$\left(\frac{\partial F}{\partial x^1}\right)_{\phi(p)} \left[\frac{dx^1(\gamma(s))}{ds}\right]_{s=0} + \ldots + \left(\frac{\partial F}{\partial x^n}\right)_{\phi(p)} \left[\frac{dx^n(\gamma(s))}{ds}\right]_{s=0}$$

$$= \sum_{\mu=1}^{n} \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} \left[\frac{dx^\mu(\gamma(s))}{ds}\right]_{s=0}$$

$$= \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} \left[\frac{dx^\mu(\gamma(s))}{ds}\right]_{s=0}$$

Einstein's summation convention

[5]without loss of generality

2.3.1 Example

Let $M = \mathbb{R}^2$ take $y = 2x^2 - 3$ to be a parabola in $\mathbb{R}^2$. We parametrise this by $x = s,\ y = 2s^2 - 3\ (\phi \circ \gamma)$

$$\phi \circ \gamma(s) = (x(s), y(s)) = (s, 2s^2 - 3)$$

$$\mathcal{F} = F(s, 2s^2 - 3)$$
$$\frac{d\mathcal{F}}{ds} = \frac{dF}{dx} \cdot 1 + \frac{dF}{dy} \cdot 4s$$
$$= \vec{T} \cdot \vec{\nabla} F$$

where $\vec{T} = [1, 4s]$
the RHS ($\vec{T} \cdot \vec{\nabla} F$) is the rate of change of F in the direction of $\vec{T}$ (might be familiar from fluid dynamics)
The map $\dot{\gamma}_p : f \to \left[\frac{d\mathcal{F}}{ds}\right]_{s=0}$ we called a tangent vector at p. We must further show that these maps live in a vector space of dimension $n = dim(M)$

Theorem 2. *The set of tangent vectors at p, $T_p(M)$, form a vector space* [6]

[6](closed under vector addition, and scalar multiplication)