MA 346, PDEs

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List of Theorems and Examples

List Of Lectures

LECTURE 1: 19/1/12

1 Introduction

1.1 References

- *Partial differnetial equations* by Evans,
- *Applied partial differential equations* by Habermann,
- *Partial differential equations, an introduction* by Strauss,

1.2 Definitions

PDE vs. ODE: A differential equation is one that involves an unknown function, u, and its derivatives. If u=u(t) depends on one variable, we denote its derivatives using primes. (u′, u″, etc.) and we call the equation ORDINARY, an ODE. If $u = u(x_1, x_2, x_3, \ldots, x_n)$ depends on $n \geq 2$ variables, we denote its derivatives using subscripts ($u_{x_1}, u_{x_1 x_2}$ etc.) and we call the equation PARTIAL, a PDE.

Order: The highest-order derivative appearing in the equation.
For instance, $u'' - u' = sin(u)$ is a 2nd-order ODE.
and $u_{xxx} - u_{yy}^2 = u$ is a 3rd-order PDE.
We'll generally omit the dependance on variables and avoid writing $u_{xxx}(x, y) - u_{yy}(x, y)^2 = u(x, y)$.

Linear: An ODE or PDE is linear if the coefficients of the unknown function u and its derivatives do not depend on either u or its derivatives. eg. $u_x + yu_y = x^2 u$ is linear (despite the x^2), $u_x^2 + yu_y = 0$ is nonlinear, $uu_y = 1$ nonlinear and so on. Second order LINEAR ODEs have the form

$$a(t)u''(t) + b(t)u'(t) + c(t) = d(t)$$

First order LINEAR PDEs in u=u(x,y):

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$$

2 Seperation of Variables

This is a method for solving some equation and finding some solutions. The idea is to look for solutions involving functions of one variable, say $u(x, y) = F(x)G(x)$ or $u(x, y, z) = F(x)G(y)H(z)$. Those are called seperable.

Example 2.1 1
Consider the PDE $u_x + xu_y = 0$, where u=u(x,y).
Take

$$u(x, y) = F(x)G(y) \quad \Rightarrow \quad F'(x)G(y) + xF(x)G'(y) = 0$$

$$\frac{F'(x)}{xF(x)} = \frac{-G'(y)}{G(y)} = \lambda$$

LHS is indep. of y and RHS is indep of x

$$\begin{aligned} F'(x) = \lambda x F(x) \quad &\& \quad G'(y) = -\lambda G(y) \\ \frac{F'(x)}{F(x)} = \lambda x \quad &\& \quad \frac{G'(y)}{G(y)} = -\lambda \\ logF(x) = \frac{\lambda x^2}{2} + c_1 \quad &\& \quad logG(y) = -\lambda y + c_2 \\ F(x) = c_3 e^{\frac{\lambda x^2}{2}} \quad &\& \quad G(y) = c_4 e^{-\lambda y} \end{aligned}$$

2.1 Transport (or Traffic Flow) Equations

This is a simple but useful PDE.
Consider the traffic flow through a one-dimensional road (or the flow of a liquid through a one-dimensional pipe)
Let u(x,t)=density of cars at point x and at time t. i.e. mass/length,
and v(x,t)=velocity of cars (overall) at point x and time t i.e.length/time.
We assume v is known and u is unknown. Then $\int_a^b u(x, t)dx$ = total number of cars in the interval [a,b].

$$\Rightarrow \frac{d}{dt}\int_a^b u(x, t)dx = -(\text{outflow at b}) + (\text{inflow at a})$$

$$= -u(b,t)v(b,t) + u(a,t)v(a,t)$$
$$= -\int_a^b [uv]_x dx$$
$$\Rightarrow \int_a^b [u_t + (uv)_x] dx = 0$$

for any a,b!!!
Thus,

$$u_t + (uv)_x = 0$$

gives the transport equation when v is known.
If we had a road that gets narrower we could choose a v function that gets smaller at those points. We can just make sure the velocity is bigger when the road has for example, more lanes or smaller when there is more turns.

Example 2.2 2
Take $v = v(t)$ for simplicity. Then we get $u_t + vu_x = 0$.
Say $u(x,t) = F(x)G(t) \Rightarrow F(x)G'(t) + vF'(x)G(t) = 0$.

$$\Rightarrow \frac{G'(t)}{G(t)} = \frac{-vF'(x)}{F(x)} = -\lambda$$
$$\frac{G'(t)}{G(t)} = \lambda v(t) \quad \& \quad \frac{F'(x)}{F(x)} = \lambda$$
$$logG(t) = \lambda \int v(t) + c_1 \quad \& \quad logF(x) = -\lambda x + c_2$$
$$G(t) = c_3 e^{\lambda \int v(t) dt} \quad \& \quad F(x) = c_4 e^{\lambda x}$$

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Example 2.3 3
Consider the PDE vs. ODE.

$$u_x + yu_y + y_z u_z = 0$$

There are seperable solutions u=F(x)G(y)H(z). Then, this becomes

$$F'GH + yFG' + yzFGH' = 0$$

$$\frac{F'}{F} + \frac{yG'}{G} + \frac{yzH'}{H} = 0$$

The part of the equation that is only dependent on x is directly related to the part that *isn't* dependent on x, then the following must be the case:

$$\frac{F'(x)}{F(x)} = -\frac{yG'(y)}{G(y)} - \frac{yzH'(z)}{H(z)} = \lambda$$

$$F'(x) = \lambda F(x) \quad \& \quad \frac{G'(y)}{G(y)} + \frac{zH'(z)}{H(z)} = -\frac{\lambda}{y}$$

Same thing again with the second equation.

$$F'(x) = \lambda F(x) \quad \& \quad \frac{H'(z)}{H(z)} = \frac{\mu}{z} \quad \& \quad \frac{G'(y)}{G(y)} = \frac{\lambda}{y} - \mu$$

Thus we have to solution:

$$logF(x) = \lambda x + c_1 \quad \& \quad logH(z) = \mu logz + c_2 \quad \& \quad logG(y) = -\lambda logy - \mu y + c_3$$

$$F(x) = Ce^{\lambda x} \quad \& \quad H(z) = cz^{\mu} \quad \& \quad G(y) = y^{-\lambda}e^{-\mu y}$$

3 Method of characteristics

This is a general method due to Hamilton, and applies to *any* first-order PDE. Consider a simple example:

$$a(x,y)u_x + b(x,y)u_y = 0$$

$$au_x + bu_y = 0$$

This looks like the chain rule.

$$x = x(s) \quad \& \quad y = y(s)$$

$$u = u(x,y) \Rightarrow \frac{du}{ds} = \frac{dx}{ds}u_x + \frac{dy}{ds}u_y$$

But these derivatives are equal to 0, a, and b respectively. We solve

$$\frac{dx}{ds} = a \quad \& \quad \frac{dy}{ds} = b \quad \& \quad \frac{du}{ds} = 0$$

Which are ODEs. The first two ODEs give a curve in the xy-plane x=x(s), y=y(s). The third equation says u is constant along the curve. We thus need only know the value of u at some point on the curve.

Example 3.1 A
We solve $u_x - u_y = 0$ subject to $u(x,0) = f(x)$.
Say $x = x(s), y = y(s) \Rightarrow \frac{du}{ds} = u_x \frac{dx}{ds} + u_y \frac{dy}{ds}$
Need to solve

$$\frac{dx}{ds} = 1,\ x(0) = x_0 \quad \& \quad \frac{dy}{ds} = -1,\ y(0) = 0 \quad \& \quad \frac{du}{ds} = 0,\ u(0) = f(x_0)$$

by writing $u(x_0, 0) = f(x_0)$.

$x = s + c_1 = s + x_0$
We get $y = -s + c_2 = -s$
$u = constant = f(x_0)$

Eliminate x_0,s(in terms of (x,y).
$s = -y, x_0 = x - s = x + y \Rightarrow u = f(x+y)$.
Charac: $x = s + x_0 = -y + x_0 \Rightarrow x + y = x_0 = constant$

Example 3.2 B
We solve $xu_x + u_y = 0$ subject to $u(x_0, 0) = f(x_0)$.
Charecteristics $\begin{pmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{du}{ds} \\ x(0) = x_0 & y(0) = 0 & u = f(x) \end{pmatrix}$. We get $\begin{pmatrix} x = ce^s = x_0 e^s \\ y = s + c = s \\ u = f(x_0) \end{pmatrix}$[1]
Eliminate x_0, s: $s = y, x_0 = xe^{-y} \Rightarrow u = f(x_0) = f(xe^{-y})$
Characteristic curves: [2]

$$xe^{-y} = x_0 = c$$

$$x = C^{-y}$$

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Example 3.3 C

$$yu_y + u_x$$

[1] $\frac{dx}{ds} = \lambda x \Rightarrow \frac{dx}{x} = \lambda ds \Rightarrow x = Ce^{\lambda s}$
[2] our charecteristic curves intersect with the real curves so we will see later what happens when they don't intersect

Subject to $u(x,0) = f(x)$. Solve

$$\frac{dx}{ds} = 1 \quad \& \quad \frac{dy}{ds} = y \quad \& \quad \frac{du}{ds} = 0$$

$$x = s + x_0 \quad \& \quad y = y_0 e^s \quad \& \quad u = u_0$$

In our case, the initial conditions give

$$x = s + x_0 \quad \& \quad y = 0 \quad \& \quad u = f(x_0)$$

So we *cannot* eliminate x_0, s, Note that the characteristic curves are

$$x = s + x_0 \quad \& \quad y = y_0 e^s \quad \Rightarrow y = y_0 e^{x - x_0} = y_0 e^{-x_0} e^x$$

We had an initial condition on $y = 0$, and that *was* a charecteristic curve!

3.1 Characteristics

Consider $au_x + bu_y = c$ with a, b, c functions of x, y, u. Let $x = x(s)$ and $y = y(s)$. Solve

$$\frac{dx}{ds} = a \quad \& \quad \frac{dy}{ds} = b \quad \& \quad \frac{du}{ds} = c \tag{3.1}$$

Subject to the initial conditions[3]

$$x(0) = x_0 \quad \& \quad y(0) = y_0 \quad \& \quad u(0) = u_0$$

Suppose a, b, c are smooth in Equation 3.4.1. By ODE theory, there exists a unique solution (defined for some s). This is defined in terms of x_0 and s, say. We wish to elimate those, so we wish to invert the transformation

$$(x, y) \to (x_0, s)$$

We can do that, provided that the Jacobian, J is nonzero on the initial curve.

$$J = det \begin{pmatrix} \frac{dx}{dx_0} & \frac{dx}{ds} \\ \frac{dy}{dy_0} & \frac{dy}{ds} \end{pmatrix}$$

This solvability condition merely says that the initial curve should not be a multiple of the characteristic curve *on the initial curve.*

$$\begin{pmatrix} \frac{dx}{dx_0} \\ \frac{dy}{dy_0} \end{pmatrix} \neq N \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

[3] this next equation can (and should) be expressed in terms of one variable, say $u(x_0, 0) = f(x_0)$ or $u(0, y) = f(y_0)$

We'll prove this in more detail (and also uniqueness).

Example 3.4 D (No Solutions)
Consider $u_x + yu_y = u$, subject to $u(x, 0) = x$. We have:

$$\frac{dx}{ds} = 1 \quad \& \quad \frac{dy}{ds} = y \quad \& \quad \frac{du}{ds} = u$$

- Checking Solvability; if the initial condition is $u(x_0, 0) = x_0$, the Jacobian is

$$J = det \begin{pmatrix} 1 & 1 \\ 0 & y \end{pmatrix} = 0 \quad \Rightarrow det \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

- Pick another initial curve , say, $u(0, y_0) = f(y_0)$. The solvability condition requires

$$J = det \begin{pmatrix} 0 & 1 \\ 1 & y \end{pmatrix} \Rightarrow det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq 0$$

  which is fine.

- Now solve:

$$u_x + yu_y = u \quad \& \quad u(0, y_0) = f(y_0)$$

  We have:

$$\frac{dx}{ds} = 1 \quad \Rightarrow \quad x = s$$
$$\frac{dy}{ds} = y \quad \Rightarrow \quad y = y_0e^s$$
$$\frac{du}{ds} = u \quad \Rightarrow \quad u = f(y_0)e^s$$

Thus $u = f(y), e^{-x}, e^x \cdots$ gives solutions to the PDE. Thus $u(x, 0) = f(0)e^x$, so we can never have $u(x, 0) = x$.

Example 3.5 E Infinitely Many Solutions
Consider

$$u_x + yu_y = u$$

as before, but impose $u(x, 0) = e^x$. Then, we can have $u = f(ye^{-x})e^x$ and $u(x, 0) = e^x$, implies $f(0) = 1$. We can take $f(z) = 1, f(z) = e^s$, or even $f(z) = z^n + 1$ etc.

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Theorem 3.6 Characteristics
Consider the first-order PDE $au_x + bu_y = c$ with a, b, c smooth functions of x, y and $u = u(x, y)$. Impose the initial conditions $x = x_0, y = y_0, u = u_0$ expressed in terms of one variable, say x_0. Let $x(s), y(s), u(s)$ be the solution of

$$\frac{dx}{ds} = a \quad \& \quad \frac{dy}{ds} = b \quad \& \quad \frac{du}{ds} = c$$

subject to the same initial conditions. If the solvability condition:

$$det \begin{pmatrix} \frac{dx}{ds} & \frac{dx}{dx_0} \\ \frac{dy}{ds} & \frac{dy}{dy_0} \end{pmatrix} \neq 0$$

holds on the initial curve, then the PDE has a unique *local* solution, obtained by inverting the transformation

$$(x, y) \to (x_0, s)$$

Proof. Solve the system of ODEs. There is a local solution, so we can express x, y, u in terms of x_0, s. By assumption and the inverse function theorem, we can solve x_0, s in terms of x, y. Define

$$u(x, y) = u(x_0(x, y), s(x, y))$$

Then

$$au_x + bu_y = a\left(\frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}\right) + b\left(\frac{\partial u}{\partial x_0}\frac{\partial x_0}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}\right)$$
$$= \frac{\partial u}{\partial x_0}\left(a\frac{\partial x_0}{\partial x} + b\frac{\partial x_0}{\partial y}\right) + \frac{\partial u}{\partial s}\left(a\frac{\partial s}{\partial x} + b\frac{\partial s}{\partial y}\right)$$

but

$$\frac{\partial u}{\partial s} = c \quad \& \quad a\frac{\partial s}{\partial x} + b\frac{\partial s}{\partial y} = 2\frac{\partial s}{\partial s} = 1 \quad \& \quad a\frac{\partial x_0}{\partial x} = \frac{\partial x_0}{\partial s} = 0$$

This proves existence of a solution.
To prove uniqueness, suppose $v(x, y)$ is another solution. Then

$$\frac{d}{ds}v(x(s), y(s)) = v_x\frac{dx}{ds} + v_y\frac{dy}{ds}$$

$$= av_x + bv_y$$
$$= c(x(s), y(s), v(x(s), y(s))$$

and we have uniqueness of solutions by ODE theory, so $v(x(s), y(s)) = u(x(s), y(s))$ $\square$

3.1.1 Remark

The method of characteristics still applies when u depends on 3 or more variable. If $u = u(x, y, z)$ depends on 3 variables and

$$au_x + bu_y + cu_z = A$$

one solves
$$\frac{dx}{ds} = a \quad \& \quad \frac{dy}{ds} = b \quad \& \quad \frac{dz}{ds} = c \quad \& \quad \frac{du}{ds}$$
The initial condition becomes $u(x, y, 0) = f(x, y)$ so it descirbes a surface.

Example 3.7 A
We'll solve

$$u_x + 2xyu_y + zu_z = u$$

Characteristic Equations:

$$\frac{dx}{ds} = 1 \quad \& \quad \frac{dy}{ds} = 2xy \quad \& \quad \frac{dz}{ds} = z \quad \& \quad \frac{du}{ds} = u$$

[4] We get
$$x = s + x_0 \quad \& \quad z = z_0 + e^s \quad \& \quad u = u_0 e^s$$
Then we can solve
$$\frac{dy}{ds} = 2xy = 2(s + x_0)y$$
$$\frac{dy}{y} = (2s + 2x_0)ds$$
$$\log y = s^2 + 2x_0 s + \log y_0$$
$$y = e^{s^2} e^{2x_0 s} y_0$$

We avoid imposing $z_0 = 0$ or even $y_0 = 0$. Say $x_0 = 0$ and start eliminating: initial condition is $u(0, y_0, z_0) = f(y_0, z_0)$. Then we have

$$x = s \quad \& \quad y = y_0 e^{s^2} \quad \& \quad z = z_0 e^s \quad \& \quad u = u_0 e^s = f(y_0, z_0) e^x$$

[4] we can't solve $\frac{dy}{ds} = 2xy$ yet!

So finally:

$$u = f(ye{-}x^2, ze^{-x})e^x$$

This is quite a general solution to the PDE but we will do it in more generality tomorrow!

Visit http://www.maths.tcd.ie/ pete/pde2 in order to find homework ther tomorrow.

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3.2 Fully Non-Linear Case

Consider *any* 1st-order PDE in *any* number of variables. Suppose $u = u(x_1, x_2, x_3, \cdots, x_n)$ and let $\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$. A first order PDE has the form

$$F(u, \nabla u, \vec{x}) = 0$$

We've seen the case

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_n u_{x_n} = b$$

in which case we get

$$\frac{dx_i}{ds} = a_i \quad \& \quad \frac{du}{ds} = b$$

We now treat the general case

$$F(\vec{x}, u, \nabla u) = 0$$

Write $p = \nabla u$ forconvenience[5]: $p_i = u_{x_i}$

3.3 Derivation of the Characteristic equations

We need to determine $x_i(s), u(s), p_i(s)$. The most dificult term is:

$$\frac{dp_i}{ds} = \sum_{j=1}^{n} \frac{\partial p_i}{\partial x_j} \frac{\partial x_j}{\partial s} = \sum_{j=1}^{n} u_{x_i x_j} x_j'$$

$u_{x_i x_j} = \frac{\partial u_{x_j}}{\partial x_i}$ is not a term we want. We will try to eliminate it. (This derivation could be on the exam). Differentiate the PDE with respect to x_i:

$$F_{x_i} + F_u u_{x_i} + \sum_{j=1}^{n} F_{u_{x_j}} \frac{\partial u_{x_j}}{\partial x_i} = 0$$

[5] p here would be momentum

Suppose

$$x_i'(s) = F_{p_i} \tag{3.2}$$

$$u'(s) = \sum_{i=1}^{n} p_i F_{p_i} \tag{3.3}$$

$$p_i'(s) = -F_{x_i} - p_i F_u \tag{3.4}$$

For each $1 \leq i \leq n$. This is a *closed* system of $2n+1$ ODEs that we can solve as before.

3.3.1 Special Case 1

Consider the PDE of the form

$$\sum_{i=1}^{n} a_i u_{x_i} = b$$

This gives

$$F = \sum a_i p_i - b$$

Char equations:

$$\text{(3.2)} \quad \frac{dx_i}{ds} = a_i$$

$$\text{(3.3)} \quad \frac{du}{ds} = \sum a_i p_i = b$$

If a_i, b are functions of x_i and u, this is a *closed* system. So we don't need Eq. 3.4.

3.3.2 Special Case 2

Consider a Hamilton-Jacobi equation, namely a PDE of the form

$$u_t + H(\vec{x}, \nabla u) = 0$$

where

$$\vec{x} = (x_1, x_2, \cdots, x_n)$$

$$u = u(x_1, x_2, \cdots, x_n)$$

$$\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$$

Think of t as x_{n+1}.
<u>Characteristic equations</u>

$$F = p_{n+1} + H(x_1, \cdots, x_n, p_{n+1})$$

$$\text{(3.2)} \quad x_i' = H_{p_i} \quad \& \quad t' = 1 \quad for 1 \leq i \leq n$$

$$\text{(3.4)} \quad p_i' = -H_{x_i} \quad \& \quad p_{n+1}' = 0 \quad for 1 \leq i \leq n$$

The only nontrivial ODEs are

$$\left.\begin{array}{ll} \text{(3.2)} & x_i' = H_{p_i} \\ \text{(3.4)} & p_i' = -H_{x_i} \end{array}\right\} \text{Hamilton's Equations}$$

Once we have solved this system, u can be obtained be

$$\text{(3.3)} \quad u' = \sum_{i=1}^{n} p_i F_{p_i} + p_{n+1}$$

upon integration.

3.3.3 Special case 3

Suppose

$$u = u(x, y)$$

and let $p = u_x$, In that case

$$\text{(3.2)} \quad x' = F_p \quad \& \quad y' = F_q$$

$$\text{(3.3)} \quad u' = pF_p + qF_q$$

$$\text{(3.4)} \quad p' = -F_x - pF_u \quad \& \quad q' = -F_y - qF_u$$

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where $u = u(x, y)$, $p = u_x$, $q = u_y$ and $F(x, y, u, p, q) = 0$

Example 3.8 A
We solve

$$u_x^2 u_y = 1$$

subject to $u(x, 0) = x$
<u>Characteristics</u> $\rightarrow$ $F = p^2 q - 1 = 0$

$$x' = 2pq \quad \& \quad y' = p^2$$

$$u' = 3p^2 q = 3 \quad \Rightarrow \quad u = 3s + u_0 = 3s + x_0$$

$$p' = 0 \quad \Rightarrow \quad p = p_0$$

$$q' = 0 \quad \Rightarrow \quad q = q_0$$

Thus

$$\begin{aligned} x &= 2p_0q_0s + x_0 \\ y &= p_0^2 + y_0 \\ u &= 3s + x_0 \\ p &= p_0 \\ q &= q_0 \end{aligned}$$

- We have to eliminate x_0, p_0, q_0, s
We can determine p_0, q_0 from the initial condition and the PDE. Since $u(x, 0) = x$ we get $u_x(x, 0) = 1$ so $p_0 = 1$.
We can get q_0 from the PDE.

$$p_0^2 q_0 = 1 \quad \Rightarrow \quad q_0 = 1$$

- Putting everything together

$$\left\{ \begin{array}{c} x = 2s + x_0 \\ y = s \\ u = 3s + x_0 \end{array} \right\} \quad \Rightarrow \quad u = 3y + (x - 2y) = x + y$$

3.4 Heat (or diffusion) Equation

- A 2nd-order PDE that describes heat propogation ansd also diffusion
- Heat propogation through a 1-dim object (a rod) is described by

$$u_t - ku_{xx} = 0$$

where $u(x, t)$ =temperature at point x at time t, and $k > 0$ a constant.

- For 2-dim objects (a membrane), we get:

$$u_t - ku_{xx} - ku_{yy} = 0$$

where $u = u(x, y, t)$.

- For n-dim objects:

$$u_t - k\Delta u = 0$$

where $u = u(x_1, x_2, \cdots, x_n, t)$ and $\Delta u \sum_i u_{x_i x_j}$

- Diffusion is described by the same PDE with u =concentration of the dissolving subtance at point $\vec{x}$ at time t.

3.5 Derivation of the heat/diffusion equation

- We need the basic facts

  1. **Fourier's Law**: heat flows from hot to cold at a rate which is proportional to the temperature difference. Note the differnce between heat & temperature
  2. Heat is proportional to temperature times mass

- Let u be the temperature. Let ϕ be the *heat flux*. The derivation is easier in 1-d.
  Then 1 says
  $$\phi = -c_1 u_x$$
  and 2 says
  $$\text{Heat} = c_2 m u$$

- Consider a small portion of the rod with constant cross section A and uniform density ρ. Then
  $$\text{mass} = \rho A\, dx$$
  $$\Rightarrow \quad \text{heat} = \rho A u(x,t)\, dx$$
  Thus the heat contained in the whole rod is
  $$H = \int_a^b \rho A u\, dx$$

- Therefore,
  $$H_t = \int_a^b c_2 \rho A u_t\, dx$$
  but also
  $$H_t = (\text{heat flowing inside}) - (\text{heat going outside})$$
  $$= \phi(a,t) - \phi(a,b)$$
  $$= -c_1 u_x(a,t) + c_1 u_x(b,t)$$
  $$= \int_a^b c_1 u_{xx}(x,t)\ dx$$
  This gives $u_t = k u_{xx}$

- In n-dimensions 1 becomes $\phi - c_1 \nabla u$ and 2 remains the same

$$\text{Heat} = \int_A c_2 \rho u \, dV$$

$$H_t = \int c_2 \rho u_t \, dV$$

$$H_t = \int_{\partial\phi} \vec{\phi}.\vec{n} \, dS$$

$$H_t = -\int c_1 \nabla u.\vec{n} \, dS = c_1 \int div(\nabla u) \, dV$$

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4 Boundary Value Problems

Consider the heat equation

$$u_t - k u_{xx} = 0$$

that describes the temperature of a metal rod, say. We impose some initial condition $u(x, 0) = f(x)$ where $0 \le x \le L$ and $L =$Length of rod. In this case, the initial condition *does not* uniquely determine the soltuion: We have to specify conditions at the endpoints

DIRICHLET conditions: we get to specify the value of u on the boundary

NEUMANN conditions: we get to specify the value of the derivative of u on the boundary. For n-dimensional problems that would be the normal derivative

$$\nabla u \cdot \vec{n}$$

PERIODIC conditions: this is for the 1d-case

4.1 DIRICHLET PROBLEM for the heat equation in [0,L]

This problem can be described as

$$u_t - k u_{xx} = 0 \quad \text{for} \quad 0 \le x \le L, t \ge 0$$

$$u(x,0) = f(x) \quad \text{for} \quad 0 \le x \le L$$
$$u(0,t) = a(t) \quad \text{for} \quad t \ge 0$$
$$u(L,t) = b(t) \quad \text{for} \quad t \ge 0$$

Step 1

We'll use seperation of variables to find some solutions. Take $a(t) = 0 = b(t)$, zero Dirichlet conditions. Take

$$u(x,t) = F(x)G(t)$$

Need

$$u_t = ku_{xx}$$
$$F(x)G'(t) = kF''(x)G(t)$$
$$\Rightarrow \frac{G'(t)}{kG(t)} = \frac{F''(x)}{F(x)} = -\lambda$$

where λ is a constant. We get

$$G'(t) = -k\lambda G(t) \quad \Rightarrow \quad G(t) = c_1 e^{-k\lambda t}$$
$$F''(x) = -\lambda F(x)$$

but Zero Dirichlet condtions

$$\to F(0) = 0 = F(L)$$

We have to solve the problem

$$\left\{ \begin{array}{l} F''(x) = -\lambda F(x) \\ F(0) = 0 = F(L) \end{array} \right\}$$

Let's look for nonzero solutions!! Such solutions exist only for some λ. We solve $F''(x) + \lambda F(x) = 0$

1. If $\lambda = 0$, we get $F(x) = c_1 + c_2 x$. Since $F(0) = 0, c_1 = 0$. Then

$$F(L) = c_2 L \quad \Rightarrow \quad c_2 = 0 \quad \Rightarrow \quad F = 0$$

2. If $\lambda > 0$, say $\lambda = a^2$, we get $F'' + a^2 F = 0$
so

$$F(x) = c_1 \sin(ax) + c_2 \cos(ax)$$

Since $F(0) = 0$, we get $c_2 = 0$. Then

$$0 = F(L) = c_1 \sin(aL)$$

and we need $aL = n\pi$ for some integer n. That is

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad \& \quad F(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

Lemma 4.1 1
The problem

$$\left\{ \begin{array}{c} F''(x) = -\lambda F(x) \\ F(0) = 0 = F(L) \end{array} \right\}$$

has nonzero solutions only when $\lambda = \left(\frac{n\pi}{L}\right)^2$ for some positive integer n, in which case

$$F(x) = c_1 \sin\left(\frac{n\pi x}{L}\right)$$

Lemma 4.2 2
The heat equation $u_t - ku_{xx} = 0$ has solutions

$$u(x,t) = c \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Those satisfy $u(0,t) = 0 = u(L,t)$ and also $\lim_{t\to\infty} u(x,t) = 0$ for all x.

Lemma 4.3 3
Suppose

$$\left\{ \begin{array}{c} F''(x) = -\lambda F(x) \\ F(0) = 0 = F(L) \end{array} \right\}$$

and $F(x)$ nonzero. Then $\lambda > 0$

Proof. Multiply the ODE by $F(x)$:

$$\int_0^L F(x)F''(x)\, dx = -\lambda \int_0^L F(x)^2\, dx$$

$$\Rightarrow \int_0^L F'(x)^2\, dx = \lambda \int F(x)^2\, dx$$

□

In particular,

$$u(x,t) = c \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

satisfies the PDE and the boundary conditions! We have to now worry about the initial condition.
Step 2
Find more general solutions. Since $u_t - ku_{xx} = 0$ is linear, the sum of two solutions is a solution:

$$\left\{ \begin{array}{c} u_t - ku_{xx} = 0 \\ v_t - kv_{xx} = 0 \\ w = u + v \end{array} \right\} \quad \Rightarrow \quad w_t - kw_{xx} = 0$$

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Similarly, any scalar multiple of a solution is a solution. This means:[6]

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

is still a solution of the PDE that satisfies the *boundary* conditions. The initial conditions requires

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

which is the Fourier sine series for $f(x)$.

Step 3

We determine the Fourier coefficients a_n. We use the formulas

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right)\, dx = 0 \tag{4.1}$$

whenever $n \neq m$.

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{L}{2} \tag{4.2}$$

Assuming those, the coefficients a_n can be found as follows:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad f(x)\sin\left(\frac{k\pi x}{L}\right) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right)$$

$$\Rightarrow \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right)\, dx = a_k \frac{L}{2}$$

Thus the only possible solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

with

$$a_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

[6]should remember this formula until June!!

4.1.1 Remark

Equation (4.1) holds because the functions $f(x) = \sin\left(\frac{n\pi x}{L}\right)$ solve the problem

$$f_n''(x) = -\left(\frac{n\pi}{L}\right)^2 f_n(x)$$

say

$$f_n''(x) = -\lambda_n f_n(x)$$

Equation (4.1) says $\{f_n\}$ are orthogonal. For vectors $\vec{x}, \vec{y}, \ldots$ orthogonality $\sum x_i y_i = 0$
For functions $f, g \ldots$ orthogonality $\int fg = 0$

Theorem 4.4 Orthogonality
Suppose

$$f_m'' = -\lambda_m f_m$$
$$f_n'' = -\lambda_n f_n$$

with $\lambda_n \neq \lambda_m$. Suppose f_m, f_n vanish at $0, L$. Then

$$\int_0^L f_m(x) f_n(x)\, dx = 0$$

Proof. We have

$$f_n f_m'' = -\lambda_m f_m f_n$$

and

$$f_m f_n'' = -\lambda_n f_m f_n$$

so

$$\int_0^L f_n f_m'' - f_m f_n'' = (\lambda_n - \lambda_m) \int f_m f_n$$

The left hand side is

$$[f_n f_m' - f_m f_n']_0^L = 0$$

□

Theorem 4.5 Uniqueness of Solutions
Consider the most general problem associated with the heat equation

$$u_t - k u_{xx} = H(x,t)$$

$$u(x,0) = f(x)$$

$$u(0,t) = a(t)$$
$$u(L,t) = b(t)$$

Then at most one solution exists.

Proof. To check the uniquness, suppose u and v are both solutions. We need to show $w = u - v$ is zero. Note that

$$w_t - kw_{xx} = 0$$

$$w(x,0) = 0$$
$$w(0,t) = 0$$
$$w(L,t) = 0$$

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Let

$$G(t) = \int_0^L w(x,t)^2 \, dx$$

Then

$$G'(t) = \int_0^L 2ww_t \, dx = \int_0^L 2kww_{xx} \, dx$$
$$G'(t) = [2kww_x]_0^L - \int_0^L 2kw_x^2 \, dx$$

so G(t) decreases, but $G(0) = \int_0^L w(x,0)^2 \, dx = 0$. Since $G(t) \geq 0$, we get $G = 0$, so $w = 0$. □

[7]

Example 4.6 Zero Dirichlet conditions 1

We solve

$$u_t - ku_{xx} = 0$$

subject to zero Dirichlet conditions and[8]

$$u(x,0) = 2\sin\left(\frac{\pi x}{L}\right) - 3\sin\left(\frac{3\pi x}{L}\right)$$

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[7]this kind of proof could very well be asked on the exam, at least the idea of proving uniqueness etc.
[8]note that we could say that $L = \pi$ to make our equations nicer to write if we want!

We know the solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

where the coefficient a_n is chosen such that:

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

Thus $a_1 = 2, a_3 = -3$, all others $a_n = 0$. Thus

$$u(x,t) = 2\sin\left(\frac{\pi x}{L}\right) e^{-k\left(\frac{\pi}{L}\right)^2 t} - 3\sin\left(\frac{3\pi x}{L}\right) e^{-9k\left(\frac{\pi}{L}\right)^2 t}$$

Example 4.7 Zero Dirichlet conditions 2

We solve

$$u_t - ku_{xx}$$

subject to zero Dirichlet conditions and subject to $u(x,0) = x$
The solution is given by the usual formula with

$$a_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) x \, dx$$

Integrate by parts:

$$a_n = -\left[\frac{2L}{n\pi L}\cos\left(\frac{n\pi x}{L}\right) x\right]_{x=0}^{L} + \frac{2L}{n\pi L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) \; dx$$

So

$$a_n = -\frac{2L}{n\pi L}\cos(n\pi)L + \frac{2L^2}{(n\pi)^2 L}\left[\sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$

Since $\sin(n\pi) = 0$, we get

$$a_n = -\frac{2L}{n\pi}\cos(n\pi) = -\frac{2L}{n\pi}(-1)^n$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

4.2 Non-homogeneuos case, heat equation, zero Dirichlet

- We solve

$$u_t - ku_{xx} = H(x,t)$$
$$u(x,0) = f(x)$$
$$u(0,t) = 0$$
$$u(L,t) = 0$$

- Decompose as

$$v_t - kv_{xx} = 0$$
$$v(x,0) = f(x)$$
$$v(0,t) = 0$$
$$v(L,t) = 0$$

  plus

$$w_t - kw_{xx} = H$$
$$w(x,0) = 0$$
$$w(0,t) = 0$$
$$w(L,t) = 0$$

  The first problem is the same as the one we had before. We need to solve the non-homogeneous one.

- Look at the Fourier series:

$$f(x) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

  obtained for each time t. We have to determine $a_n(t)$.

We'll relate this Fourier series to those of

$$w_t = \sum b_n(t) \sin\left(\frac{n\pi x}{L}\right)$$
$$w_{xx} = \sum c_n(t) \sin\left(\frac{n\pi x}{L}\right)$$
$$H = \sum d_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

Then

$$b_n - kc_n = d_n$$

by the PDE. also

$$b_n = \frac{2}{L}\int_0^L w_t \sin\left(\frac{n\pi x}{L}\right)\, dx$$
$$= \frac{d}{dt}\left[\frac{2}{L}\int_0^L w \sin\left(\frac{n\pi x}{L}\right)\, dx\right]$$
$$\Rightarrow \quad b_n = a_n'$$
$$\Rightarrow \quad a_n' - kc_n = d_n$$

now a and c are unknown but d is known.

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Step 1

First, we solve the homogeneuos problem with *Dirichlet* conditions

$$u_t - ku_{xx} = 0 \quad \& \quad u(x,0) = f(x) \quad \& \quad u(0,t) = 0 = u(L,t)$$

We did this using seperation of variables:

$$u = F(x)G(t) \quad \Rightarrow \quad \frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)} = -\lambda$$

So we end up with

$$\left\{ \begin{array}{c} F''(x) + \lambda F(x) = 0 \\ F(0) = 0 = F(L) \end{array} \right\}$$

This gives $\lambda = \left(\frac{n\pi}{L}\right)^2$ and $F(x) = c\sin\left(\frac{n\pi x}{L}\right)$ and the solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

provided that

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

namely when

$$a_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

Step 2

Next, we solve the non-homogeneous

$$u_t - ku_{xx} = H(x,t) \quad \& \quad u(x,0) = f(x)$$

with *zero* Dirichlet conditions

$$u(0,t) = 0 = u(L,t)$$

Let v be the solution to the homogeous problem 1.

$$u_t - ku_{xx} = H \quad \& \quad u(x,0) = 0$$

$$v_t - kv_{xx} = 0 \quad \& \quad v(x,0) = f(x)$$

Then $w = u + v$ satisfies

$$w_t - kw_{xx} = H \quad \& \quad w(x,0) = f(x)$$

namely the whole problem. We have to solve

$$\left\{ \begin{array}{c} u_t - ku_{xx} = H(x,t) \\ u(x,0) = 0 \\ u(0,t) = u(L,t) = 0 \end{array} \right\}$$

Suppose the solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

We have to determine the coefficients a_n. We'll relate those to the coefficients of u_t, u_{xx}, H denoted by $b_n.c_n, d_n$ respectively. First of all

$$b_n = \frac{2}{L} \int \sin\left(\frac{n\pi x}{L}\right) u_t \; dx = a_n'$$

$$c_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) u_{xx} \; dx$$

$$c_n = \left[\frac{2}{L} \sin\left(\frac{n\pi x}{L}\right) u_x\right]_0^L - \frac{2}{L} \int \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right) u_x \; dx$$

$$c_n = \left[\frac{2}{L} \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right) u\right]_0^L - \frac{2}{L} \int_0^L \left(\frac{n\pi}{L}\right)^2 \sin\left(\frac{n\pi x}{L}\right) u$$

$$c_n = -\left(\frac{n\pi}{L}\right)^2 a_n$$

$d_n = b_n - kc_n$ by the PDE

$$\Rightarrow \quad a_n' + k\left(\frac{n\pi}{L}\right)^2 a_n = d_n$$

This is a first order linear!! We can solve using integrating factors:

$$\left[a_n e^{k\left(\frac{n\pi}{L}\right)^2 t}\right]' = d_n e^{k\left(\frac{n\pi}{L}\right)^2 t}$$

$$a_n(t) = \int_0^t d_n(s) e^{k\left(\frac{n\pi}{L}\right)^2 (s-t)}\, ds$$

Step 3
We'll Solve the general problem

$$\left\{\begin{array}{c} u_t - k u_{xx} = H \\ u(x,0) = f(x) \\ u(0,t) = a(t) \\ u(L,t) = b(t) \end{array}\right\}$$

The new part is the presence of nonzero Dirichlet conditions. The idea is to pick ANY function that satisfies them, say

$$v(x,t) = a(t) + \frac{x}{L}[b(t) - a(t)]$$

This is some function that satisfies *only* the boundary condtions. Let $w = u - v$. Then

$$w_t - k w_{xx} = (u_t - k u_{xx}) - (v_t - k v_{xx}) = H - v_t$$

$$w(x,0) = f(x) - v(x,0)$$

$$w(0,t) = 0 = w(L,t)$$

we know how to solve this problem.[9]

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4.3 Fourier Series Facts

4.3.1 Full Fourier series

Suppose f is (piecewise) smooth on $[-L, L]$. Then

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$$

for some unique coefficients a_n, b_n. The series converges to $f(x)$ at points where f is continuous and to the average value at the other points. This makes the Fourier series periodic with period $2L$.

[9] $H - v_t$ and $f(x) - v(x,0)$ are known functions

4.3.2 Fourier Sine Series

Suppose f is piecewise smooth on $[0, L]$. Then

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$a_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx$$

The series converges to $f(x)$ at all points where f is continuous and to 0 at the endpoints (for any f). *USEFUL FOR DIRICHLET*

4.3.3 Fourier Cosine Series

We have

$$f(x) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right)$$

with

$$b_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx \quad \text{for} n \geq 1$$

and

$$b_0 = \frac{1}{L}\int_0^L f(x)\, dx$$

The derivative of this series will vanish at the endpoints. *USEFUL FOR NEUMANN*

4.3.4 WARNING

Let $f_n(x) = \frac{x}{(1+x^2)^n}$. This is differentiable for all x, however

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \begin{cases} 0 & \text{if} x = 0 \\ \frac{1+x^2}{x} & \text{if} x \neq 0 \end{cases}$$

so the sum is not even continuous

- For power (Taylor) series

$$f(x) = \sum_{n=0}^{\infty} a_n x_n$$

one can differentiate term by term (in the interval of convergence)

- For Fourier series

$$f(x) = \sum a_n \sin\left(\frac{n\pi x}{L}\right)$$

this is fine if $f(0) = 0 = f(L)$. For Fourier cosine series, it's always fine.

- For functions of two variables

$$u(x,t) = \sum a_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

one <u>has to know</u> something about $a_n(t)$ in order to differentiate[10]

For example

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

$$u_t(x,t) = \sum a_n'(t) \sin\left(\frac{n\pi x}{L}\right) \quad ???$$

to expand u_t using Fouries, the coefficients are

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) u_t \, dx$$

$$b_n = \frac{d}{dt}\left[\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) u \, dx\right] = a_n'(t)$$

4.4 Heat Equation, Neumann Conditions

Consider the problem

$$u_t - ku_{xx} = 0$$

$$u(x,0) = f(x)$$

$$u_x(0,t) = 0 = u_x(L,t)$$

Look for seperable solutions

$$u = F(x)G(t) \quad \Rightarrow \quad \frac{G'(t)}{kG(t)} = \frac{F''(x)}{F(x)} = -\lambda$$

Thus

$$F''(x) = -\lambda F(x) \quad \& \quad G'(t) = -k\lambda G(t)$$

[10] an infintie sum is really just a limit. $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$. Thus when we want to diffentiate infinite series what we are really asking is does the limit commute with the derivative, this is not always true!

Lemma 4.8 **λ and F solution**
Suppose F is a non-zero solution of

$$F''(x) + \lambda F(x) = 0$$

$$F'(0) = 0 = F'(L)$$

Then $\lambda = \left(\frac{n\pi}{L}\right)^2$ with $n \geq 0$ and

$$F(x) = a\cos\left(\frac{n\pi x}{L}\right)$$

Proof.

$$F'' + \lambda F = 0$$

implies

$$\lambda F^2 = -FF''$$

$$\Rightarrow \lambda \int F(x)^2 = -\int FF'' = [-FF']_0^L + \int (F')^2$$

$$[-FF']_0^L = 0 \text{ by Dirichlet/Neumann/periodic}$$

This shows $\lambda \geq 0$. We get

$$\lambda = 0 \quad \Rightarrow \quad F(x) = Ax + B$$

$A = 0$ because $F'(L) = 0$

$$\lambda > 0 \quad \Rightarrow \quad \lambda = a^2$$

$$\Rightarrow \quad F(x) = c_1 \sin(ax) + c_2 \cos(ax)$$

$c_1 = 0$ becasuse $F'(0) = 0$ Also

$$F'(x) = -c_2 \sin(ax)$$

$$\Rightarrow \quad a = \frac{n\pi}{L} \quad \text{because} \quad F'(L) = 0$$

□

Thus

$$u(x,t) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

so anything we did for the Dsirichlet condition, we can do for the Neumann!.

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4.5 Wave Equation

We now look at the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

that describes the propogation of waves and also the motion of a guitar string. This is second-order in time so we have to specify <u>two</u> initial conditions:

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= H(x,t) \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= a(t) \\ u(L,t) &= b(t) \end{aligned}$$

4.5.1 Uniqueness of Solutions

Suppose u, v solutions. Then $w = u - v$ satisifes

$$w_{tt} - c^2 w_{xx} = 0$$

with zero initial and boundary conditions. Consider

$$E(t) = \frac{1}{2}\int w_t^2 + \frac{c^2}{2}\int_0^L w_x^2$$

Then

$$E'(t) = \int_0^L w_t w_{tt} + c^2 w_x w_{xt}\, dx$$

$$E'(t) = \int_0^L c^2 w_t w_{xx} + [c^2 w_x w_t]_0^L - \int c^2 w_{xx} w_t$$

The boundary terms are zero assuming <u>Neumann</u> $w_x = 0$ but also Dirichlet $\left\{ \begin{array}{c} w(0,t) = 0 \\ w(L,t) = 0 \end{array} \right\} \quad \Rightarrow \quad w_t = 0$ at endpoints. So

$$E(t) = E(0) = 0 \quad \Rightarrow \quad w_t = w_x = 0 \text{ at all points}$$

$\Rightarrow w$ is constant, hence 0.

Example 4.9 wave equation
We'll solve

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(x,0) = 0$$
$$u_t(x,0) = g(x)$$
$$u_x(0,t) = u_x(L,t) = 0$$

Seperation of Variables

$$u = F(x)G(t)$$
$$-\lambda = \frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)}$$
$$F''(x) = -\lambda F(x) \quad \& \quad G'' = -\lambda c^2 G$$
$$F'(0) = 0 \quad \& \quad G(0) = 0$$
$$F'(L) = 0$$

The first problem gives

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n \geq 0$$
$$F(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$$

And then

$$G''(t) + \left(\frac{n\pi c}{L}\right)^2 G(t) = 0$$
$$\Rightarrow G(t) = c_2 \sin\left(\frac{n\pi ct}{L}\right)$$

Thus

$$u(x,t) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right)$$

and we need $u_t(x.0) = g(x)$. Let's find the Fouries cosine series of u_t:

$$\frac{2}{L}\int_0^L u_t \cos\left(\frac{n\pi x}{L}\right)\, dx = \frac{d}{dt}\left[\int_0^L u \cos\left(\frac{n\pi x}{L}\right)\right]$$

$$\Rightarrow \quad \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right) = \frac{d}{dt}\left[b_n \sin\left(\frac{n\pi ct}{L}\right)\right] = \frac{n\pi c}{L} b_n \cos\left(\frac{n\pi ct}{L}\right)$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right) \quad \text{if } n \geq 1 \ (n = 0 \text{ is irrelevant})$$

4.6 Laplace Equation over a triangle

The equation is $\Delta u = 0$ or simply

$$u_{xx} + u_{yy} = 0$$

It descirbes the time-independent solution of $u_t - k\Delta u = 0$, heat equation, or $u_{tt} - c^2\Delta u$, wave.
Let's solve this *linear* PDE over a rectangle $0 \leq x \leq L, 0 \leq y \leq M$. There are boundary conditions on the four sides

$$picture$$

Seperation of variables

$$u = F(x)G(y)$$
$$-\lambda = \frac{F''}{F} = -\frac{G''}{G}$$
$$F'' = -\lambda F \quad \& \quad G'' = \lambda G$$
$$F(L) = 0 \quad \& \quad G(M) = 0, G(0) = 0$$

so

$$\lambda = -\left(\frac{n\pi}{M}\right)^2$$
$$G(y) = c_1 \sin\left(\frac{n\pi x}{M}\right)$$

and then

$$F'' - \left(\frac{n\pi}{M}\right)^2 F = 0$$
$$F(x) = c_1 \sinh\left(\frac{n\pi(x-L)}{M}\right)$$

LECTURE 13: Thursday 16 February

5 Eigenvalue Problems

A typical problem is

$$F'' + \lambda F = 0$$

One has to find eigenvalues λ for which non-zero solutions (eigenfunctions) F exist when we impose boundary conditions, either Dirichlet or Neumann. A generalization of this when $u = u(x_1, x_2, \cdots, x_n)$ and

$$-\Delta u = \lambda u \text{ in some set A}$$

$$u = 0 \text{ on the boundary } \partial A$$

$$\text{or } \frac{\partial u}{\partial n} = 0 \text{ on the boundary } \partial A$$

We've seen the special case $A = [0, L]$ in the interval $\mathbb{R}$. Under Dirichlet conditions

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \text{ with } n \geq 1$$

and

$$u_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Moreover, these eigenfunctions are orthogonal in the sense that

$$\int_0^L u_m(x) u_n(x)\, dx = 0$$

and <u>smooth</u> function $f(x) = \sum c_n u_n(x)$. Our goal is to show that we <u>always</u> have a sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

of eigenvalues with $\lambda \to \infty$ as $n \to \infty$, eigenfunctions with distinct eigenvalues are orthogonal and any smooth function can be expressed as

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x)$$

$u_n(x)$ being the eigenfunction.

5.1 Calculus Facts

One has the divergence theorem

$$\int_A \nabla \vec{F} = \int_{\partial A} \vec{F} \vec{n}$$

with $\vec{F}$ a vector and $\vec{n}$ the unit normal vector to A, and

$$\nabla \vec{F} = F_{x_1} + F_{x_2} + \cdots$$

We'll need this fact in the case that $\vec{F} = u.\nabla v$ for scalar functions u, v. In this case

$$div\vec{F} = \sum_k (F^k)_{x_k}$$

$$div\vec{F} = \sum_k (uv_{x_k})_{x_k}$$

$$div\vec{F} = \sum_k u_{x_k}v_{x_k} + uv_{x_kx_k}$$

$$div\vec{F} = \nabla u \nabla v + u\Delta v$$

We get

$$\int_A \nabla u \nabla v + u\Delta v = \int_{\partial A} u\nabla v\vec{n}$$

We'll generally write $\frac{\partial u}{\partial n}$ =normal derivative of $u = \nabla u.\vec{n}$

$$\int_A u\Delta v = -\int_A \nabla u\nabla v + \int_{\partial A} u\frac{\partial v}{\partial n} \tag{5.1}$$

$$\int_A u\Delta v - v\Delta u = \int_{\partial A} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \tag{5.2}$$

Theorem 5.1 1
Consider the eigenvalue problem

$$-\Delta u = \lambda u$$

in A. Then the eigenvalues are positive, if one requires $u = 0$ on ∂A, and non-negative, if $\frac{\partial u}{\partial n} = 0$ on ∂A

Proof. Suppose $-\Delta u = \lambda u$ and u is non-zero. Then

$$-\int_A u\Delta u = \lambda\int_A u^2$$

and we can use (5.1) to get

$$\lambda\int_A u^2 = -\int_A u\Delta u = \int_A |\nabla u|^2 - \int_{\partial A} u\frac{\partial u}{\partial n}$$

but $\int_{\partial A} u\frac{\partial u}{\partial n} = 0$. Thus

$$\lambda = \frac{\int |\nabla u|^2}{\int u^2} \geq 0$$

and if $\lambda = 0$ then u is constant. □

Theorem 5.2 2
Eigenfunctions that correspond to distinct eigenvalues are orthogonal:

$$\int_A u(x)v(x) = 0$$

if

$$-\Delta u = \lambda u$$
$$-\Delta v = \mu v$$
$$\lambda \neq \mu$$

Moreover, the gradients $\nabla u, \nabla v$ are also orthogonal.

Proof. We have

$$-v\Delta u = \lambda uv$$

and

$$-u\Delta v = \mu uv$$

so

$$\int_A u\Delta v - v\Delta u = (\lambda - \mu)\int_A uv$$
$$\Rightarrow \quad \int_{\partial A} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} = (\lambda - \mu)\int_A uv$$
$$\Rightarrow \quad \int_A uv = 0$$

Moreover,[11]

$$0 = \int_A u\Delta v = -\int_A \nabla u \nabla v - \int_{\partial A} u\frac{\partial v}{\partial n}$$

□

LECTURE 14: 17th February 10am

Consider the eigenvalue problem

$$-\Delta u = \lambda u$$

in some bounded subset $A \in R^n$ subject to <u>Dirichlet</u> conditions $u = 0$ on ∂A. Note that

$$\lambda \int_A u^2 = -\int_A u\Delta u = \int_A |\nabla u|^2 - \int_{\partial A} u\frac{\partial u}{\partial n}$$

[11]boundary terms will be zero by either neumann or dirichlet

$$\Rightarrow \quad \lambda = \frac{\int_A |\nabla u|^2}{\int_A u^2}$$

We call the right hand side the RAYLEIGH QUOTIENT

$$R(u) = \frac{\int_A |\nabla u|^2}{\int_A u^2}$$

If u is an eigenfunction, then $R(u) = \lambda$ is the corresponding eigenvalue. We'll show that the eigenvalues form a sequence $\lambda_1 \leq \lambda_2 \leq \cdots$, that the first eigenfunctions u_1 is a function that minimizes $R(u)$, the second u_s minimizes $R(u)$ over all $u \perp u_1$ nd so on.

Example 5.3 Eigenvalue problem
Take $A = [0,1] \in R$ and Dirichlet conditions. Then $\lambda_n = (n\pi)^2 = n^2\pi^2$ and the least eigenvalue is π^2. The eigenfunctions are $u_n = \sin(n\pi x)$. The claim above implies

$$\pi^2 \leq \frac{\int u'(x)^2}{\int u(x)^2}$$

for all functions with $u(0) = 0 = u(1)$. Thus, there is no function u with

$$\int_0^1 u'(x)^2 = 9 \int_0^1 u(x)^2$$

with $u(0) = 0 = u(1)$.

Theorem 5.4 3, First eigenfunction and eigenvalue
Suppose that $R(u)$ has a minimum, say $m = \min R(u)$ over all functions in

$$X = \{u \in C^2(A) : \ u = 0 \ on \ \partial A \ and \ u \neq 0\}$$

Then m is an eigenvalue, it is the least eigenvalue and any function that minimizes $R(u)$ is an eigenfunction.

We'll give two proofs based on the same idea; the derivative should be zero at points of min/max.

Definition 5.5 First Variation or Frechet derivative
Suppose $I(u)$ is a functional that depends on a function u. Then the first variation of $I(u)$ or the Frechet derivative in the direction of ϕ is

$$I'(u)\phi = \lim_{\epsilon \to 0} \frac{I(u + \epsilon\phi) - I(u)}{\epsilon}$$

Example 5.6 I(u) example
Let $I(u) = \int_A |\nabla u(x)|^2$. Then

$$I(u + \epsilon\phi) = \int |\nabla u + \epsilon \nabla \phi|^2$$

$$= \int (\nabla u + \epsilon\nabla\phi)(\nabla u + \epsilon\nabla\phi)$$

$$= \int |\nabla u|^2 + 2\epsilon \nabla u \nabla \phi + \epsilon^2 |\nabla \phi|^2$$

so

$$I'(u)\phi = \int 2\nabla u \nabla \phi$$

Example 5.7 another I(u)
Let $I(u) = \int u^2 \, dx$ Then

$$I(u + \epsilon\phi) = \int u^2 + 2\epsilon\phi u + \epsilon^2\phi^2$$

so

$$I'(u)\phi = \int 2u\phi \, dx$$

Proof. of theorem 3
We are assuming $R(u)$ has a minimum value m attained at w, say. Then $R'(w)\phi = 0$ for any function ϕ. We could use the quotient rule but we won't.
By above, the function

$$f(\epsilon) = R(w + \epsilon\phi)$$
$$= \frac{\int |\nabla w + \epsilon\nabla\phi|^2}{\int (w + \epsilon\phi)^2}$$

a function of one variable!!. Namely

$$f(\epsilon) = \frac{\int |\nabla w|^2 + 2\epsilon \int \nabla w \nabla \phi + \epsilon^2 \int |\nabla\phi|^2}{\int w^2 + 2\epsilon w\phi + \epsilon^2\phi^2}$$

That's

$$f(\epsilon) = \frac{a_1 + b_1\epsilon + c_1\epsilon^2}{a_2 + b_2\epsilon + c_2\epsilon^2} = \frac{N(\epsilon)}{D(\epsilon)}$$

Then $f'(0) = 0$ by calculus

$$\Rightarrow \quad 0 = f'(0) = \frac{N'(0)D(0) - D'(0)N(0)}{D(0)^2}$$

$$\Rightarrow \quad b_1 a_2 - b_2 a_1 = 0$$
$$\Rightarrow \quad (2\int \nabla w \nabla \phi)(\int w^2) - (\int 2w\phi)(\int |\nabla w|^2) = 0$$
$$\Rightarrow \quad \int \nabla w \nabla \phi - \int w\phi R(w) = 0$$
$$\Rightarrow \quad \int \nabla w \nabla \phi - mw\phi = 0$$

for all $\phi \in X$. Integrate by parts to get

$$\int_A (-\Delta w - mw)\phi = 0$$

for all $\phi \in X$. This implies

$$-\Delta w - mw = 0$$
$$-\Delta w = mw$$

so w is an eigenfunction, m is an eigenvalue. It's also the least since $R(u) = \lambda$ for eigenfunction u □

LECTURE 15: 17th February 3pm

We gave one proof using calculus by looking at $f(z) = r(u + \epsilon\phi)$

Proof. Second proof using Frechet derivatives
Note that $R(cu) = R(u)$ for any constant c. This means we can look for a minimizer with $\int u^2 = 1$. This is a constrained minimzation problem $I(u) = \int_A |\nabla u|^2$ subject to the constant

$$J(u) = \int_A u^2 = 1$$

By Lagrange multipliers, one has

$$I'(u)\phi = \lambda J'(u)\phi$$

for all directions ϕ. Thus

$$\int_A 2\nabla u \nabla \phi = \lambda \int_A 2u\phi$$
$$\int_A -\Delta u\phi = \int_A \lambda u\phi$$

for all ϕ.

$$-\Delta u = \lambda u$$

□

Theorem 5.8 4, Characterization of the nth eigenvalue
Suppose that $R(u)$ attains a minimum over the set

$$X_n = \{u \in C^2(A) : u = 0 \text{ on } \partial A \; u \neq 0 \;\&\; u \perp u_1, \cdots, u_{n-1}\}$$

Then any functions that minimizes $R(u)$ over X_n is an eigenfunction u_n with eigenvalue

$$\lambda_n = \min_{u \in X_n} R(u)$$

and obviously perpendicular to $u_1, \cdots, u_{n-1}$ and obviously $\lambda_{n-1} \leq \lambda_n$.

Proof. Suppose that $m = \min_{u \in X_n} R(u)$ exists. Then $f(\epsilon) = R(w + \epsilon\phi)$ becomes min when $\epsilon = 0$ so $f(0) = 0$ and we get

$$-\int_A \Delta w \phi = \int_A m w \phi$$

exactly as before, but only for $\phi \in X_n$. This does not automatically imply $-\Delta w = mw$. We now claim this equation holds for <u>any</u> function $\psi \in X_1$. Write

$$\psi = \psi_1 + \psi_2$$

with $\psi_1 \perp u_1, u_2, \cdots, u_{n-1}$. Namely [12]

$$\psi_2 = \sum_{k=1}^{n-1} c_k u_k$$

$$\psi_1 = \psi - \sum_{k=1}^{n-1} c_k u_k$$

Since

$$\int \psi_1 u_i = \int \psi u_i - \sum_{k=1}^{n-1} c_k \int u_k u_i$$

we need

$$0 = \int \psi u_i - c_i \int u_i^2$$

$$\Rightarrow \quad c_i = \frac{\int \psi u_i}{\int u_i^2}$$

for all i.
We know that

$$-\int_A \Delta w \phi = \int_A m w \phi$$

[12] Need $\psi_1 \perp u_i$ for all i

holds for ψ_1. Thus it remains to check it for $\psi_2 = \sum_{k=1}^{n-1} c_k u_k$. We need to check

$$-\int_A \Delta w \psi_2 = \int_A m w \psi_2$$

or simply

$$-\int_A \Delta w u_k = \int_A m w u_k \tag{5.3}$$

Integrate by parts to move the Laplacian onto u_k:

$$\int_A u\Delta v - v\Delta u = \int_{\partial A} u\frac{\partial v}{\partial \vec{n}} - v\frac{\partial u}{\partial \vec{n}}$$

This reduces (5.3) to

$$\Leftrightarrow \quad -\int w\Delta u_k = \int m w u_k$$

$$\Leftrightarrow \quad \int w\lambda_k u_k = \int m w u_k$$

$$\Leftrightarrow \quad (\lambda_k - m)\int w u_k = 0$$

This holds since $w \in X_n$ □

We now know $\lambda_1 \leq \lambda_2 \leq \cdots$ assuming that $\min_{u\in X_n} R(u)$ exists. We'll check this and also completeness: any smooth function can be expanded as

$$u(x) = \sum_{n=1}^{\infty} c_n u_n(x)$$

LECTURE 16: Thursday 23rd February

We have assumed that the Rayleigh quotient $R(u) = \frac{\int_A |\nabla u|^2}{\int_A u^2}$ attains a mimimum over $X\{u \in C^2 : u = 0 \text{ on} \partial A,\ u \neq 0\}$. If that is true, the minimum λ_1 is the first eigenvalue and similarly λ_n is the minimum of $R(u)$ over $X_n\{u \in C^2 : u = 0 \text{ on } \partial A,\ u \neq 0 \text{ and } u \perp u_1, \cdots, u_{n-1}\}$ assuming this exists. This gives a sequence of eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

and we'll show $\lambda \to \infty$ as $n \to \infty$. Assuming this, one can easily show there is no other eigenvalue $\lambda_n < \lambda < \lambda_{n+1}$.

To show that the minima above exist, we need some facts about
L^2, the set of u with $\int_A u^2$ finite
H^1, the Sobolev space, the set of all u with $\int_A u^2, \int_A |\nabla u|^2$ finite. Here $\int |\nabla u|^2 = \int u_{x_1}^2 + \cdots + u_{x_n}^2$.

5.2 The Main Idea

We need to minimize

$$R(u) = \frac{\int |\nabla u|^2}{\int u^2}$$

over all nonzero functions in H^1. Since $R(cu) = R(u)$, this is the same as minimizing

$$\{\int |\nabla u|^2 : \int u^2 = 1\}$$

Start with

$$\inf_{u \in H^1}\{\int |\nabla u|^2 : \int u^2 = 1\}$$

and try to show that the inf is attained. Since d =inf of a set, we have a sequence that converges to it. There is a function u_n in the set we're minimizing over such that

$$d \leq \int |\nabla u_n|^2 \leq d + \frac{1}{n} \tag{5.4}$$

$$\int u_n^2 = 1 \tag{5.5}$$

We are hoping that u_n converges to some function u with

$$\int |\nabla u|^2 = d$$

$$\int u^2 = 1$$

Such a function is a minimizer!!!

Theorem 5.9 A, Rellich's Compactness Theorem
Suppose $A \subset \mathbb{R}^n$ is bounded. Let u_n be a sequence of functions which are bounded in H^1:

$$\int u_n^2 + \int |\nabla u_n|^2 \leq C$$

for all n and some constant C. Then there is a subsequence u_{n_k}(to be denoted by u_n) that converges to some function $\tilde{u}$ in L^2:

$$\lim \int |u_n - \tilde{u}|^2 = 0 \tag{5.6}$$

Theorem 5.10 B, Banach-Alaoglu
Suppse A, u_n are as before. Then there is a subsequence that converges weakly in H^1:

$$\lim_{n\to\infty} \int (u_n - u)\varphi = 0 \tag{5.7}$$

$$\lim_{n\to\infty} \int \nabla(u_n - u)\nabla\varphi = 0 \tag{5.8}$$

for any functiontion $\varphi \in H^1$.

Step 1
These two limits are the same:

$$\int (u_n - \tilde{u})^2 \to 0 \ \ (5.6)$$

and

$$\int (u_n - u)\varphi \to 0 \ \ (5.7)$$

imply $u = \tilde{u}$. Namely

$$|\int (u_n - \tilde{u})\varphi| \le ||u_n - \tilde{u}||_2 ||\varphi||_2 \to \ 0$$

for any φ. Thus

$$\int (u_n - \tilde{u})\varphi \ \to \ 0$$

$$\int (u_n - u)\varphi \ \to \ 0$$

$$\Rightarrow \ \int_A \tilde{u}\varphi = \int_A u\varphi$$

for all $\varphi \in H^1$
Step 2
Let u_n be the minimizing sequence:

$$\int u_n^2 = 1 \quad \& \quad d \le \int |\nabla u_n|^2 \le d + \frac{1}{n}$$

Then $\int u^2 = 1$ because (5.6) implies

$$\int u_n^2 - 2uu_n + u^2 \ \to \ 0$$

so

$$\int u_n^2 - 2uu + u^2 \ \to \ 0 \ \text{ by (5.7)}$$

so

$$\int u_n^2 \;\rightarrow\; \int u^2$$

Thus $u \in H^1$ and $\int u^2 = 1$ so

$$d = \inf\{|\nabla u|^2 : \int u^2 = 1\} \quad \& \quad d \leq \int |\nabla u|^2$$

We'll show $\int |\nabla u|^2 \geq d$.

LECTURE 17: Friday 24th February 10am

5.3 Existence of a minimum

Let

$$d = \inf\left\{ \int_a |\nabla u|^2 \;:\; \int_A u^2 = 1, u = 0 \text{ on } \partial A \right\}$$

We automatically get a sequence $u_n \in H^1$ with

$$\int u^2 = 1 \quad \& \quad d \leq \int |\nabla u|^2 \leq d + \frac{1}{n} \quad \& \quad u_n = 0 \text{ on } \partial A$$

Then we get (using the hard theorems) a subsequence that we'll denote by u_n such that

$$\int (u_n - u)^2 \rightarrow 0 \quad \& \quad \int u_n \phi \rightarrow \int u\phi \quad \& \quad \int \nabla u_n \nabla \phi \rightarrow \int \nabla u \nabla \phi$$

for some function $u \in H^1$ that vanishes on ∂A and all $\phi \in H^1$.
The first of those limits gives *strong* convergence in L^2. The second is *weak* convergence.

$$\int (u_n - u)\phi \rightarrow 0$$

Note that

$$\left| \int (u_n - u)\phi \right| \leq ||u_n - u||_2 \; ||\phi||_2$$

by Holder's
Using the first, we get

$$\int u_n^2 - 2uu_n + u^2 \rightarrow 0 \quad \& \quad \int u_n^2 - u^2 \rightarrow 0$$

$$1 = \lim \int u_n^2 = \int u^2$$

Therefore

$$\int |\nabla u|^2 \geq d$$

by definition of d. We'll show

$$\int |\nabla u|^2 \leq d$$

In fact,

$$\int |\nabla u|^2 = \int \nabla u \nabla u = \lim_{n\to\infty} \int \nabla u_n \nabla u = \lim_{n\to\infty} \inf \int \nabla u_n \nabla u$$
$$\leq \lim_{n\to\infty} \inf ||\nabla u_n||_2 \, ||\nabla u||_2$$
$$= \sqrt{d} \, ||\nabla u||_2$$

and so

$$||\nabla u||^2 \leq \sqrt{d} \, ||\nabla u||$$

Which implies that either $||\nabla u|| = 0$ or $||\nabla u|| \leq \sqrt{d}$. In the latter case

$$\int |\nabla u|^2 \leq d$$

In the former case, u is a constant which is equal to 0 (by Dirichlet), and that's a contradiction, since $\int u_2 = 1$.

Corollary 5.11 $\lambda_n \to \infty$
One has $\lambda_n \to \infty$.

Proof. By construction λ_n is the min value of $R(u)$ over all functions orthogonal to $u_1, \cdots, u_{n-1}$ (the corresponding eigenfunctions). Thus λ_n is *increasing*. Suppose $\lambda_n \leq L$, and that u_n are orthonormal

$$u_n^2 = 1$$

Then

$$\int |\nabla u|^2 = R(u_n) = \lambda_n$$

and $u_n^2 = 1$. By Rellich's compactness theorem, there is a subsequence

$$\int (u_n - u)^2 \to 0$$

for some $u \in H^2$. Then

$$\int (u_n - u_m)^2 \to 0$$

for large m, n.

$$\int (u_n - u_m)^2 = \int u_n^2 - 2u_m u_n + u_m^2 = 2$$

which is a contradiction. □

Theorem 5.12 Completeness
Suppose $\{u_n\}$ are the eigenfunctions for the eigenstate problem $-\Delta u = \lambda u$. Then every $f \in L^2$ can be written as

$$f = \sum_{n=1}^{\infty} c_n u_n$$

for some unique c_n. More precisely, one has

$$R_N = f - \sum_{n=1}^{N} c_n u_n$$

which satisfies $\int R_N^2 \to 0$.

Proof. We'll prove this for some C^2 functions that vanish on ∂A. The general case follows by approximating $f \in L^2$ by such functions. Assume $\{u_n\}$ orthonormal. We claim that

$$R_N \perp u_1, \cdots, u_N \tag{5.9}$$

in the sense that $\int R_N u_k = 0$, and

$$\nabla R_N \perp \nabla u_1, \cdots, \nabla u_N \tag{5.10}$$

in the sense that $\int \nabla R_N \nabla u_k = 0$.
To check this, let $1 \leq k \leq N$. Then

$$R_N u_k = \int u_k f - \sum_{n=1}^{N} c_n \int u_n u_k = u_k f - c_k \ \cdots \text{by orthonormality} = 0$$

because $c_k = \int u_k f$ by definition. Similarly

$$\int \nabla R_n \nabla u_k = \int \nabla u_k \nabla f - \sum_{n=1}^{N} c_n \int \nabla u_n \nabla u_j = -\int \Delta u_k f + \sum_{n=1}^{N} c_n \Delta u_n u_k$$

as $-\Delta u_k = \lambda_k u_k$. Thus

$$\cdots = \int \lambda_k u_k f - \sum_{n=1}^{N} \lambda_n c_n u_n u_k = \lambda_k \int u_k f - \lambda_k c_k = 0$$

We will now use equations (5.9) and (5.10) show that

$$\int R_N^2 \to 0$$

By equation (5.9), the Rayleigh quotient

$$\frac{\int |\nabla R_N|^2}{\int R_N^2} \geq \lambda_{N+1} \quad \Rightarrow \quad \int |\nabla R_N|^2 \geq \lambda_{N+1} \int R_N^2$$

and it suffices to show that the lefthand side is bounded. However,

$$\int |\nabla f|^2 = \int |\nabla R_N + \sum c_n \nabla u_n|^2 = \int |\nabla R_N|^2 + |\sum c_n \nabla u_n|^2 \geq \int |\nabla R_N|^2$$

□

LECTURE 18: Firday 24th February 3pm

6 The Hard Theorems

We'll give a sketch of their proofs. Theorem B (Banach Alaoglu) claims that a sequence $u_n \in H^1$ with $\int u_n^2, \int |\nabla u_n|^2$ bounded has a subsequence with

$$\int u_n \varphi \;\to\; \int u\varphi \quad \& \quad \int \nabla u_n \varphi \;\to\; \nabla u \nabla \varphi$$

for some function $u \in H^1$ and all $\varphi \in H^1$.

Proof. (sketch) The main point is that H^1, just like L^2, has a countable dense subdet D. This also implies that every orthonormal basis is countable: say $\{v_x\}$ is orthonormal. Then

$$\int (v_\alpha - v_\beta) = \int v_\alpha^2 - v_\beta^2 = 2$$

and each ball of radius $\frac{1}{2}\sqrt{2}$ with center in D covers the space by density and each ball contains at most one v_α.
Now, take a countable orthonormal basis $\{w_n\}_{n=1}^{\infty}$ and consider the bounded sequence u_n. Then

$$\int u_n w_1 \leq ||u_n||\,||w_1|| < \infty$$

and that's a sequence of <u>reals</u>. There is a subsequence u_n^1 such that $\int u_n^2 w_1$ converges. Similarly $\int u_n^2 w_1, \int u_n^2 w_2$ both converge for some subsequence u_n^2. We get instinctively

$$\int u_n^k w_i \text{ converges}$$

for $1 \leq i \leq k$. Then

$$\int u_n^n w_i \text{ converges}$$

for all i (any fixed i)
We now show $\int u_n^n \varphi$ is convergent for any $\varphi \in H^1$. To see this, write

$$\varphi = \sum c_i w_i \quad \& \quad c_i = \int w_i \varphi$$

Then

$$\int \varphi^2 = \int \sum_{i.j} c_i w_i c_j w_j = \sum_i c_i^2$$

so $c \ \rightarrow \ 0$
We show $\int u_n^n \varphi$ is CAUCHY. Write

$$\varphi = \sum_{i=1}^{N} c_i w_i + \sum_{i=N+1}^{\infty} c_i w_i$$

$$\varphi = \sum_{i=1}^{N} c_i w_i + R_N$$

then

$$\int R_N^2 = \sum_{i=N+1}^{\infty} c_i^2 \ \rightarrow \ 0$$

and

$$\int (u_n^n - u_m^m \varphi = \int (u_n^n - u_m^m) R_N + \sum_{i=1}^{N} \int (u_n - u_m^m) c_i w_i$$

The first term is bounded by

$$||u_n^n - u_m^m|| \, ||R_N||^2 \leq 2C ||R_N||^2 \ \rightarrow \ 0$$

The second term is bounded by

$$\max(c_1, \cdots, c_n) | \sum \int (u_n^n - u_m^m) w_i | \ \rightarrow \ 0$$

□

Theorem A (Rellich's Compactness Theorem)
If $\int u_n^2, \int |\nabla u_n|^2$ are bounded, then a subsequence satisfies

$$\int (u_n - u)^2 \ \to \ 0$$

for some $u \in L^2$

We'll only prove that when $A \subset \mathbb{R}$, say $A = [0, 1]$.

Proof. First, we use Theorem B to get

$$\int_0^1 u_n \varphi \ \to \ \int_0^1 u \varphi$$

$$\int_0^2 u_n' \varphi' \ \to \ \int_0^1 u' \varphi'$$

where $'$ denotes differentiation, for all $\varphi \in H^1$. We note that

$$u_n(x) - u_n(y) = \int_y^x u_n'(z)$$

converges to (first term by fundamental theorem, second by banach alaoulu)

$$\int u(x) - u(y) = \int u'(z)$$

We also have

$$\int [u_n(x) - u_n(y)] \varphi(x) \ \to \ \int [u(x) - u(y)] \varphi(x)$$

by Lebesgue's theorem [13]

$$\int u_n(x) \varphi(x) \ \to \ \int u(x) \varphi(x)$$

by above

$$u_n(y) \int \varphi(x) \ \to \ u(y) \int \varphi(x)$$

$$u_n(y) \ \to \ u(y) \text{ for all y}$$

Finally we claim

$$\sup_{0<x<1} |u_n(x) - u(x) \leq \epsilon$$

[13]integral of limit is the same as the integral of the limit for certain conditions

for large n. Suppose not, then $\exists\, x_n : x_n \;\rightarrow\; x$ up to a subequence and

$$\epsilon < |u_n(x_n) - u(x_n)|$$

$$\epsilon \leq |u_n(x_n) - u_n(x)| + |u_n(x) - u_n(x_n)| + |u_n(x_n) - u(x_n)|$$

The last term goes to 0 by above. [14]

$$u_n(x_n) - u_n(x) = \int_x^{x_n} u_n 1 \leq ||u_n'|| \sqrt{(x_n - x)^2} \;\rightarrow\; 0$$

□

LECTURE 19: 8th March 3pm

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7 Calculus of Variations

The general idea is the following. If a function u of one (or more) variables minimizes/maximizes a functional, then it satisfies an ODE (or a PDE).
For instance, consider

$$J(u) = \int_A |\nabla u(x)|^2 \, dx$$

where $A \subset \mathbb{R}^n$ and subject to $u(x) = f(x)$ on ∂A.

Suppose that $J(u_*)$ is the min value over all functions. Given *any* function φ that vanishes on the boundary,

$$J(u_* + \epsilon\varphi) - J(u_*) \geq 0$$

for all ϵ (because $u_* + \epsilon\varphi = f$ on ∂A).
so

$$\lim_{\epsilon \to 0} \frac{J(u_* + \epsilon\varphi) - J(u_*)}{\epsilon} = 0$$

This means $J'(u_*)\varphi = 0$ for all functions φ with $\varphi = 0$ on ∂A.

In our case

$$J(u + \epsilon\varphi) = \int |\nabla u + \epsilon \nabla\varphi|^2$$

---

[14] the line is possibly not correct

$$= \int |\nabla u|^2 + 2\epsilon \nabla u \nabla \varphi + \epsilon^2 |\varphi|^2$$

and so

$$0 = J'(u)\varphi = \lim_{\epsilon \to 0} \int 2\nabla u \nabla \varphi + \epsilon \int |\varphi|^2$$

$$= \int 2\nabla u \nabla \varphi = -\int 2\Delta u \varphi$$

for all φ, hence $\Delta u = 0$.

7.1 General Case

Consider a functional $J(y)$ that depends on a function $y(t)$ of variable, say

$$J(y) = \int_a^b L(t, y(t), y'(t))\, dt$$

for some given function L, called the Lagrangian.

Example 7.1 if $y(t)$ minimizes then we get an ODE
Take $J(y) = \int_a^b y(t)^3 + y'(t)^4\, dt$. If $y(t)$ minimizes $J(y)$, then

$$0 = J'(y)\varphi$$

$$= \lim_{\epsilon \to 0} \frac{J(y + \epsilon\varphi) - J(y)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int (y + \epsilon\varphi)^3 - y^3 + (y' + \epsilon\varphi')^4 - (y')^4$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int 3y^2 \epsilon\varphi + [\epsilon^2, \epsilon^3, \text{ terms}] + \frac{1}{\epsilon} \int 4(y')^3 \epsilon\varphi + [\epsilon^2, \epsilon^3, \epsilon^4, \text{ terms}]$$

Therefore

$$0 = \int 3y^2\varphi + 4(y')^3\varphi'(t)\, dt$$

$$= \int [3y^2 - 4\frac{d}{dt}(y')^3]\varphi\, dt$$

for all φ and so

$$3y^2 - 4 \times 3(y')^3 y'' = 0$$

a second order ODE.

Theorem 7.2 1
Suppose y is a local min or max of

$$J(y) = \int_a^b L(t, y(t), y'(t))\; dt$$

where $y(t)$ is twice continuously differentiable. Then

$$0 = J'(y)\varphi$$
$$= \int_a^b \frac{\partial L}{\partial y}\varphi + \frac{\partial L}{\partial y'}\varphi'(t)\; dt$$

for all φ with $\varphi = 0$ on ∂A.
In particular $y(t)$ is a solution of the Euler-Lagrange equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}$$

or simply

$$\frac{d}{dt}L_{y'} = L_y$$

Note
$\frac{d}{dt}L$ contains 3 terms and $\frac{\partial L}{\partial t}$ only one!

Proof. We have

$$J(y) = \int L(t, y(t), y'(t))\; dt$$

If y(t) is a local min/max, then

$$0 = J'(y)\varphi = \lim_{\epsilon \to 0} \frac{J(y + \epsilon\varphi) - J(y)}{\epsilon}$$

We have to compute

$$L(t, y + \epsilon\varphi, y' + \epsilon\varphi') - L(t, y, y')$$

Let's use Taylor series:

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \cdots$$

In our case this gives

$$L(t, y + \epsilon\varphi, y' + \epsilon\varphi') - L(t, y, y') =$$

$$= \frac{\partial L}{\partial y}(\epsilon\varphi) + \frac{\partial L}{\partial y'}(\epsilon\varphi') + [\epsilon^2 \text{ terms etc.}]$$

so

$$J'(y)\varphi = \int \frac{\partial L}{\partial y}\varphi + \frac{\partial L}{\partial y'}\varphi'$$

$$= \int \left(\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial y'}\right)\varphi = 0$$

This is true for all φ so

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial y'}$$

□

LECTURE 20: Friday 9th March 10am

Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)$$

$$f(x,y) = \sum_{n=0}^{\infty} \frac{\partial_x^n f(x_0, y)}{n!}(x - x_0)^n$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\partial_x^n \partial_y^m f(x_0, y_0)}{m!n!}(x - x_0)^n (y - y_0)^m$$

$f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) +$ higher order terms

We started with a local min/max of

$$J(y) = \int_a^b L(t, y(t), y'(t))\ dt$$

and we ended up with

$$0 = J'(y)\varphi = \int_a^b \frac{\partial L}{\partial y}\varphi + \frac{\partial L}{\partial y'}\varphi'\ dt$$

$$= \int_a^b \left[\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial y'}\right]\varphi\ dt$$

for all φ vanishing at a, b.

Lemma 7.3 Fundamental Lemma of Variational Calculus
Suppose $\int_a^b H(t)\varphi(t)\ dt = 0$ for all φ vanishing at a, b. If H is continuous, then $H = 0$.

Proof. Take

$$\varphi(t) = (b-t)(t-a)H(t)$$

Then

$$\int_a^b (b-t)(t-a)H(t)^2\,dt = 0$$
$$\Rightarrow \quad (b-t)(t-a)H(t) = 0$$

$\Rightarrow \quad H = 0$ on $[a, b]$ by continuity. □

Example 7.4 1, Standard
Consider the graphs passing through $y(a) = y_0$ and $y(b) = y_1$. The shortest path betweeen these points is a line. We have to minimize the arclength

$$J(y) = \int_a^b \sqrt{1 + y'(t)^2}\,dt$$

Over all functions $y(t)$ with $y(a) = y_0, y(b) = y_1$.
The boundary condtions do not affect the problem: if $y(t)$ is a local min/max then

$$0 = J'(y)\varphi = \lim_{\epsilon \to 0} \frac{J(y + \epsilon\varphi) - J(y)}{\epsilon}$$

since $y + \epsilon\varphi$ satisfies the boundary condtions.
We get

$$\frac{d}{dt}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}$$

with $L = \sqrt{1 + (y')^2}$.
Since $\frac{\partial L}{\partial y} = 0$, we find

$$\frac{\partial L}{\partial y'} = C$$
$$\frac{2y'}{2\sqrt{1 + (y')^2}} = C$$
$$\frac{1 + (y')^2}{(y')^2} = C$$
$$(y')^2 = C$$
$$\Rightarrow \quad y' = C$$

$\Rightarrow \quad y'$ is a line!!

Definition 7.5 y **an extremal**
We say y is an extremal (or critical point) for

$$J(y) = \int_a^b L\ dt$$

if

$$\frac{d}{dt}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}$$

Example 7.6
We find the extremals of

$$J(y) = \int_0^1 [(y')^2 - y^2 + 2ty]\ dt$$

subject to $y(0) = y(1) = 0$
In this case

$$\frac{d}{dt}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}$$

$$2y'' = 2t - 2y$$

So

$$y'' + y(t) = t$$

Which is a second order, linear, non-homogeneous ODE!
Homogeneous solution...

$$y'' + y = 0$$

$$y = c_1 \sin t + c_2 \cos t$$

Particular solution...

$$y = At + B \quad \Rightarrow \quad y = t$$

$$y = c_1 \sin t + c_2 \cos t + t$$

We need

$$y(0) = 0 \quad \Rightarrow \quad c_2 = 0$$

$$y(1) = 0 \quad \Rightarrow \quad c_1 \sin 1 + 1 = 0$$

$$\Rightarrow \quad y = t - \frac{\sin t}{\sin 1}$$

Theorem 7.7 Several unknowns
Suppose

$$J(y_1, \cdots, y_n) = \int_a^b L(t, y_1(t), \cdots, y_n(t), y_1'(t), \cdots, y_n'(t))\ dt$$

Then every extremal satisfies

$$\frac{d}{dt}\frac{\partial L}{\partial y_k'} = \frac{\partial L}{\partial y_k}$$

for each $1 \leq k \leq n$.

Proof. Consider $y_2, \cdots .y_n$ as constant. Viewing J as a function of y_1, we get

$$\frac{d}{dt}\frac{\partial L}{\partial y_1'} = \frac{\partial L}{\partial y_1}$$

Repeat. □

Example 7.8 3
Consider

$$J(y, z) = \int_a^b [y'(t)z'(t) + y(t)^2]\, dt$$

Then

$$\frac{d}{dt}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y} \quad \& \quad \frac{d}{dt}\frac{\partial L}{\partial z'} = \frac{\partial L}{\partial z}$$

$$z'' = 2y \quad \& \quad y'' = 0$$

So

$$y = c_1 t + c_2 \quad \& \quad z'' = 2y = 2c_1 t 2c_2$$

$$z = \frac{c_1 t^3}{3} + c_2 t^2 + c_3 t + c_4$$

LECTURE 21: Friday, 9th March, 3pm

Example 7.9 Catenary
A chain hung (Pete wrote hanged, but this is more correct!) between two points. What shape does it attain?
It's center of mass is

$$\frac{\int_a^b y\sqrt{1 + y'(t)^2}\, dt}{\int_a^b \sqrt{1 + y'(t)^2}\, dt}$$

The denominator = length of chain = constant.
The chain attains the most stable position, so it basically minimizes its center of mass.
We need to minimize

$$J(y) = \int_a^b y\sqrt{1 + y'(t)^2}\, dt$$

subject to boundary conditions $y(a) = y_0, y(b) = y_1$
Euler Langrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y}$$

Note that $z = y'\frac{\partial L}{\partial y'} - L$ is constant because

$$z' = y''\frac{\partial L}{\partial y'} + y'\frac{\partial L}{\partial y} - \frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y'}y'' = 0$$

Say $y'\frac{\partial L}{\partial y'} - L = c$

$$y'\frac{y2y'}{2\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = c$$

$$\frac{(y')^2}{\sqrt{1+(y')^2}} - \sqrt{1+(y')^2} = \frac{c}{y}$$

$$\frac{(y')^2 - 1 - (y')^2}{\sqrt{1+(y')^2}} = \frac{c}{y}$$

$$\Rightarrow \quad (y')^2 = \frac{y^2}{c^2} - 1$$

Hard way...

$$\frac{dy}{dt} = \pm\sqrt{\frac{y^2}{c^2} - 1}$$

is seperable
Easy way... differentiation gives $2y'y'' = \frac{2yy'}{c^2}$

$$\Rightarrow \quad y'' - \frac{1}{c^2}y = 0$$

$$\Rightarrow \quad y = A\sinh(\frac{t}{c}) + B\cosh(\frac{t}{c})$$

If we require $y(0) = 0$, for instance, we get

$$y = A\sinh(\frac{t}{c})$$

as the only extremals.

7.2 Minimization with other Constraints

As an in'cent'ive to read my notes. I'll give you 1cent if you ask for it...
Suppose we want to find extremals of

$$J(y) = \int_a^b y(t)\ dt$$

subject to $\int_a^b \sqrt{1 + y'(t)}\ dt = \pi d^2$. That is, maximize the area subject to boundary conditions and also $\int M(t, y(t), y'(t))\ dt = c$
Then $J'(y)\varphi$ may <u>not</u> be zero anymore because

$$J'(y)\varphi = \lim_{\epsilon \to 0} \frac{J(y + \epsilon\varphi) - J(y)}{\epsilon}$$

and $y + \epsilon\varphi$ may <u>not</u> satisy the new constraint (so the numerator has <u>no</u> (definite) sign.

When minimizing $f(x, y)$ subject to a constraint $g(x, y) = c$ one can use Lagrange multipliers:

$$\nabla f = \lambda \nabla g$$

unless $\nabla g = 0$ at all points.[15]

Theorem 7.10 Two minimization conditions
Suppose the Lagrangians L, M are twice continuously differentiable. Then any extremal of

$$J(y) = \int_a^b L(t, y(t), y'(t))\ dt$$

subject to

$$I(y) = \int_a^b M(t, y(t), y'(t))\ dt = c$$

being <u>constant</u> (and possibly other boundary conditions) then either:

1. $I'(y)\varphi = 0$ for all $\varphi : \varphi(a) = \varphi(b) = 0$
2. $J'(y)\varphi = \lambda I'(y)\varphi$ for all φ

Thus, y is an extremal of I of $J - \lambda I$ (for some λ).

Example 7.11 1
We'll find the extremals of

$$J(y) = \int_a^b y(t)\ dt$$

[15] the point is that everything you have for calculus you have for functional

subject to $I(y) = \int_a^b \sqrt{1+(y')^2}\; dt$ being constant <u>and</u> $y(-1) = 0 = y(1)$

<u>Extremals of I:</u>

$$\frac{d}{dt}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y} \quad \& \quad L = \sqrt{1+(y')^2}$$

$$\frac{d}{dt}\frac{y'}{\sqrt{1+(y')^2}} = 0$$

so

$$\frac{y'}{\sqrt{1+(y')^2}} = \frac{\partial L}{\partial y'} = c$$

Then

$$\frac{(y')^2}{1+(y')^2} = c$$

so

$$y' = c \quad \& \quad y = 0$$

<u>Extremals of $J = -\lambda I$</u>

$$\frac{d}{dt}\frac{\partial L}{\partial y'} = \frac{\partial L}{\partial y} \quad \& \quad L = y - \lambda\sqrt{1+(y')^2}$$

$$\frac{-\lambda y'}{\sqrt{1+(y')^2}} = \frac{\partial L}{\partial y'} = t + c$$

$$\frac{(y')^2}{1+(y')^2} = \left(\frac{t+c}{\lambda}\right)^2$$

$$\frac{1}{(y')^2} = \left(\frac{\lambda}{t+c}\right)^2 - 1$$

$$y' = \pm\frac{t+c}{\sqrt{\lambda^2-(t+c)^2}}$$

$$y = c_2 \pm \sqrt{\lambda^2-(t+c)^2}$$

$$(y-c_2)^2 + (t+c_1)^2 = \lambda^2$$

LECTURE 22: 22 March, 3pm

Theorem 7.12 Lagrange Multipliers
Suppose u is an extremal of the functions $J(u)$ subject to the constraint $I(u)$ =consant. We assume $J(u) = \int_a^b L(t, y(t), y'(t))\; dt$ with L continuously differentiable. Then <u>either</u>

- $I'(u)\varphi = 0$ for all φ and u is extremal or I.
- $J'(u)\varphi = \lambda I'(u)\varphi$ for some λ and u is extremal of $J - \lambda I$

Proof. Suppose $I'(u)\varphi \neq 0$ for some φ
We'll show

$$J'(u)\psi = \frac{J'(u)\varphi}{I'(u)\varphi} I'(u)\psi$$

For all ψ.
Suppose not, then we can find ψ:

$$I'(u)\varphi J'(u)\psi \neq J'(u)\varphi I'(u)\psi$$

Define

$$f(\epsilon, \delta) = J(u + \epsilon\varphi + \delta\psi)$$
$$g(\epsilon, \delta) = I(u + \epsilon\varphi + \delta\psi)$$

Recall

$$J'(u)\varphi = \lim_{\epsilon\to 0} \frac{J(u + \epsilon\varphi) - J(u)}{\epsilon}$$
$$J'(u + \delta\psi)\varphi = \lim_{\epsilon\to 0} \frac{J(u + \epsilon\varphi + \delta\psi) - J(u + \delta\psi)}{\epsilon}$$
$$= \lim_{\epsilon\to 0} \frac{f(\epsilon, \delta) - f(0, \delta)}{\epsilon}$$
$$= f_\epsilon(0, \delta)$$

This gives

$$J'(u)\varphi = f_\epsilon(0, 0)$$
$$J'(u)\psi = f_\delta(0, 0)$$
$$I'(u)\varphi = g_\epsilon(0, 0)$$
$$I'(u)\psi = g_\delta(0, 0)$$

Then

$$g_\epsilon f_\delta - f_\epsilon g_\delta \neq 0$$

so

$$det \begin{bmatrix} f_\delta & f_\epsilon \\ g_\delta & g_\epsilon \end{bmatrix} \neq 0$$

In particular, the transformation

$$(\epsilon, \delta) \quad \to \quad (f, g)$$

is an invertible transformation, which cannot happen at a local min/max of f (by Lagrange multipliers in $\mathbb{R}^2$ or by the inverse function theorem) □

7.3 Second Variation

The condition $J'(u)\varphi = 0$ is only necessary for having a local min/max. To check if u is really a local min/max one needs to look at the 2nd derivative.

Say $f = f(x)$ depends on one variable. Suppose x_0 is a critical point. Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

If $f'(x_0) > 0$, then we have $f(x) \geq f(x_0)$ near $x = x_0$. hence a local minimum.

If $f'(x_0) < 0$ we have a local max

If $f''(x_0) = 0$, then $f'''(x_0) = 0$ and one looks at $f''''(x_0)$ etc.

7.3.1 First variation(or derivative) of $J(u)$

Is defined as

$$J'(u)\varphi = \lim_{\epsilon \to 0} \frac{J(u + \epsilon\varphi) - J(u)}{\epsilon}$$

Thus

$$J(u + \epsilon\varphi) = J(u) + \epsilon J'(u)\varphi + \epsilon R$$

with $R \to 0$ as $\epsilon \to 0$

7.3.2 Second variation (or derivative) of $J(u)$

Is defined by the formula

$$J(u + \epsilon\varphi) = J(u) + \epsilon J'(u)\varphi + \frac{\epsilon^2}{2!}J''(u)\varphi + \epsilon^2 R$$

with $R \to 0$ as $\epsilon \to 0$. This means

$$J''(u)\varphi = \lim_{\epsilon \to 0} \frac{J(u + \epsilon\varphi) - J(u) - \epsilon J'(u)\varphi}{\frac{1}{2!}\epsilon^2}$$

Example 7.13 J(u)

Take $J(u) = \int u^2$. Then

$$J(u + \epsilon\varphi) = \int u^2 + 2\epsilon u\varphi + \epsilon^2\varphi^2$$

$$= J(u) + \epsilon J'(u)\varphi + \frac{\epsilon^2}{2}J''(u)\varphi$$

so $J'(u)\varphi = \int 2u\varphi$
and $J''(u)\varphi = \int 2\varphi^2$

Example 7.14 J
Let $J(u) = \int u^3$ Then

$$J'(u)\varphi = \int 3u^2\varphi$$

and

$$J''(u)\varphi = \int 6u\varphi^2$$

LECTURE 23: Friday 23 March 10am

Note that yesterday we used 'u' instead of 'y'. We will continue to use 'u' but the it is the same as the original 'y'

Theorem 7.15 Explicit Formula
Suppose

$$J(u) = \int_a^b L(t, u, u')\ dt$$

Then

$$J'(u)\varphi = \int_a^b L_u\varphi + L_{u'}\varphi'\ dt$$

and similarly

$$J''(u)\varphi = \int_a^b L_{uu}\varphi^2 + 2L_{uu'}\varphi\varphi' + L_{u'u'}(\varphi')^2$$

Proof. We use Taylor expansion for functions of two variables.

$$L(t, u + \epsilon\varphi, u' + \epsilon\varphi') = L(t, u, u') + \epsilon\varphi L_u + \epsilon\varphi' L_{u'} + \frac{1}{2}(\epsilon\varphi)^2 L_{uu} +$$

$$+(\epsilon\varphi)(\epsilon\varphi')L_{uu'} + \frac{1}{2}(\epsilon\varphi')^2 L_{u'u'} + \epsilon^2 R$$

with $R \to 0$ as $\epsilon \to 0$ □

Theorem 7.16 Necessary Condition for a local extremum
If $J(u)$ has a local min at the function u, then $J'(u)\varphi = 0$ and $J''(u)\varphi \geq 0$ for all φ. If u is a local max, then $J'(u)\varphi = 0 \geq J''(u)\varphi$.

Proof. We prove this for a local min.
We know $J'(u)\varphi = 0$ at a local min. Write

$$J(u+\epsilon\varphi) = J(u) + \epsilon J'(u)\varphi + \frac{\epsilon^2}{2}J''(u)\varphi + \epsilon^2 R$$

Suppose $J''(u)\varphi < 0$ for some φ. Then

$$\lim_{\epsilon\to 0}\frac{J(u+\epsilon\varphi)-J(u)}{\epsilon^2} = \frac{1}{2}J''(u)\varphi + R < 0$$

then $R \to 0$ and we have a clear contradiction. □

Theorem 7.17 Sufficient Condition for a local extremum
Suppose u is an extremal, a critical point, of $J(u)$. Suppose also that

$$J''(u)\varphi \geq \delta\left[\int_a^b \varphi^2 + \int_a^b (\varphi')^2\right]$$

for some $\delta > 0$ and all φ. Then $J(u)$ attains a local min at the function u. There is a similar conditions for a local max.

Proof. Use a Taylor expansion as before:

$$L(u, u+\epsilon\varphi, u'+\epsilon\varphi') = L(t,u,u') + \epsilon\varphi L_u + \epsilon\varphi' L_{u'} + (\frac{\epsilon^2\varphi^2}{2}L_{uu} +$$

$$+\epsilon^2\varphi\varphi' L_{uu'} + \frac{\epsilon^2(\varphi')^2}{2}L_{u'u'}) + \epsilon^2(\varphi^2 + (\varphi')^2)R$$

where $R \to 0$ as $\epsilon \to 0$.[16]

We integrate this to get

$$J(u+\epsilon\varphi) = J(u) + \frac{\epsilon^2}{2}J''(u)\varphi + \epsilon^2\int(\varphi^2 + (\varphi')^2)R$$

Thus

$$\lim_{\epsilon\to 0}\frac{J(u+\epsilon\varphi)-J(u)}{\epsilon^2} \geq (\int \varphi^2 + \int(\varphi')^2)\left(\frac{\delta}{2} + R\right)$$

with $\frac{\delta}{2} + R$ positive for small enough ϵ. □

Note that neither $J''(u)\varphi \geq 0$ nor the other condition is easy to check for <u>all</u> φ vanishing at a, b.

[16] $R + \sqrt{(x-x_0)^2 + (y-y_0)^2}$ i.e. just a distance

Theorem 7.18 Legendre necessary condtion
Suppose

$$J(u) = \int_a^b L(t, u, u')\ dt$$

Suppose u is a local minimum. Then $L_{u'u'} \geq 0$(at all points)

For instance $J(u) = \int u^2 - (u')^2$ has no local min.

Proof. We know $J''(u)\varphi \geq 0$ for all φ. This means

$$0 \leq J''(u)\varphi = \int_a^b L_{uu}\varphi^2 + 2L_{uu'}(\varphi\varphi') + L_{u'u'}(\varphi')^2$$

$$= \int_a^b (L_{uu} - \frac{d}{dt}L_{uu'})\varphi^2 + L_{u'u'}(\varphi')^2$$

$$= \int_a^b P(t)\varphi^2 + Q(t)(\varphi')^2$$

We need to show $Q(t) \geq 0$ at all points.
Suppose $Q(t_0) < 0$. Assume $a < t_0 < b$, the case $t_0 = a, b$ being similar. Then $Q(t)$ is negative on $[t_0 - \epsilon, t_0 + \epsilon]$. We take

$$\varphi(t) = \begin{cases} \sin^2\left(\frac{\pi(t-t_0)}{\epsilon}\right) & t_0 - \epsilon \leq t \leq t_0 + \epsilon \\ 0 & \text{elsewhere} \end{cases}$$

Then[17]

$$0 \leq \int_{t_0-\epsilon}^{t_0+\epsilon} P(t)\sin^4\left(\frac{\pi(t-t_0)}{\epsilon}\right)\ dt + \frac{\pi^2}{\epsilon^2}\int_{t_0-\epsilon}^{t_0+\epsilon} Q(t)\left[\sin\left(\frac{2\pi(t-t_0)}{\epsilon}\right)\right]^2\ dt$$

$$\leq max|P(t)|2\epsilon + \frac{\pi^2}{\epsilon^2}maxQ(t)\int_{t_0-\epsilon}^{t_0+\epsilon}\left[\sin\left(\frac{2\pi(t-t_0)}{\epsilon}\right)\right]^2\ dt$$

where the last integral is

$$\frac{\epsilon}{2\pi}\int_{-2\pi}^{2\pi} \sin^2 z\ dx = C\epsilon$$

This gives

$$0 \leq C_1\epsilon + \frac{C_2}{\epsilon}maxQ(t)\ \rightarrow\ \infty$$

,a contradiction. □

[17] there was originally a mistake in this but I have just removed the thing the Pete did wrong

LECTURE 24: 23 March, 3pm

Example 7.19 1
Take

$$J(u) = \int_{-1}^{1} t\sqrt{1+(u')^2}\ dt$$

Then

$$L_{u'u'} = \frac{t}{(1+(u')^2)^{\frac{3}{2}}}$$

changes sign on $[-1, 1]$ so there is no local extrema.

Example 7.20 2
Take

$$J(u) = \int_{0}^{1} \sqrt{u^2+(u')^2}$$

Then

$$L_{u'u'} = \frac{u^2}{(u^2+(u')^2)^{\frac{3}{2}}} \geq 0$$

but there are no local maxima (homework!).

7.4 Symmetries and Noether's theorem

Consider a functional like

$$J(u) = \int_{a}^{b} L(t, u, u')\ dt$$

If L has some symmetry, then there is a conserved quantity.
Ex take $J(u) = \int_a^b L(t, u')\ dt$, there is no u dependence. Replacing u by $u + \epsilon$ does not affect $J(u)$. The Euler-Lagrange equation gives

$$\frac{d}{dt}\frac{\partial L}{\partial u'} = \frac{\partial L}{\partial u}$$

and the quantity $\frac{\partial L}{\partial u'}$ is conserved (for extremals).

Example 7.21 1
Let $m, k > 0$ and

$$J(u) = \int_{a}^{b} \frac{1}{2}m(u')^2 - \frac{k}{2}u^2$$

In this case, the integral does not depends on t, we have the invariance.
Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial u'} = \frac{\partial L}{\partial u}$$

$$(mu)' = -ku$$

$$mu'' = -ku$$

Multiply by $2u'$ and integrate to get

$$m(u')^2 + ku^2 = c$$

conservation of energy.

Definition 7.22 Invariance
Consider the functional

$$J(u) = \int_a^b L(t, u(t), u'(t))\ dt$$

We say it's invariant under the transformation

$$(t, u) \quad \to \quad (t_*, u_*)$$

depending on some parameter ϵ, if

$$\int_a^b L(t, u(t), u'(t)) \ = \int_{a_*}^{b_*} L(t_*, u_*(t_*), u'_*(t_*))\ dt_*$$

and also $t_* = t, u_* = u$ when $\epsilon = 0$

Typical choices would be

$$t_* = t + \epsilon, u_* = u \quad \text{time invariance}$$

$$t_* = t, u_* = u + \epsilon \quad \text{translation invariance}$$

$$t_* = t\cos\epsilon - u\sin\epsilon, u_* = t\sin\epsilon + u\cos\epsilon \quad \text{rotational invariance}$$

Definition 7.23 Infinitesimal generators
Suppose we have a transformation

$$t_* = f(t, u)$$

$$u_* = g(t, u)$$

depending on a parameter ϵ with $t_* = t, u_* = u$ when $\epsilon = 0$. We define

$$\xi = \frac{\partial f}{\partial \epsilon}(0, 0)$$

$$\eta = \frac{\partial g}{\partial \epsilon}(0,0)$$

Theorem 7.24 Noether's Theorem
Suppose $J(u)$ is invariant under $(t,u) \quad \rightarrow \quad (t_*, u_*)$ depending on ϵ. Then

$$\eta \frac{\partial L}{\partial \epsilon} + \xi \left(L - u' \frac{\partial L}{\partial u'} \right)$$

is conserved for all extremals of the functional.

For instance, take

$$J(u) = \int \frac{1}{2} m(u')^2 - \frac{k}{2} u^2$$

No time dependance, thus

$$t_* = t + \epsilon$$

$$u_* = u$$

Then $\xi = 1, \eta = 0$ so,

$$L - u' \frac{\partial L}{\partial u'} = -\frac{1}{2} m(u')^2 - ku^2 - u' - mu'$$

but $u' - mu' = 0$

LECTURE 25: Thursday 29 March 3pm

Theorem 7.25 Noether's Theorem
Suppose the functional

$$J(u) = \int_a^b L(t, u, u')\, dt$$

is invariant under the transformation

$$t_* = f(t,u)) \quad \& \quad u_* = g(t,u)$$

depending on parameter ϵ with $t_* = t, u_* = u$ when $\epsilon = 0$. Let $\xi = \left. \frac{\partial f}{\partial \epsilon} \right|_{\epsilon=0}$ and $\eta = \left. \frac{\partial g}{\partial \epsilon} \right|_{\epsilon=0}$. Then the quantity

$$\eta \frac{\partial L}{\partial u'} + \xi \left(L - u' \frac{\partial L}{\partial u'} \right)$$

is conserved (independent of t) for each extremal u of the functional.

Proof. The proof is a somewhat messy (but straightforward) computation. Invariance means

$$\int_a^b L(t, u(t), u'(t))\ dt = \int_a^b L(t_*, u_*(t_*), u'_*(t_*))\ dt_*$$

The idea is to expand the right hand side in a Taylor series on ϵ and compare the coefficients of ϵ. We may thus ignore all terms involving ϵ^2 and higher powers of ϵ. Note that

$$t_* = f(t, u) = f(t, u)|_{\epsilon=0} + \epsilon \left.\frac{\partial f}{\partial \epsilon}\right|_{\epsilon=0} + \cdots = t + \epsilon\xi + \cdots$$

and similarly

$$u_* = u + \epsilon\eta$$

We could write $\delta t = \epsilon\xi \quad \& \quad \delta u = \epsilon\eta$.

First we expand the integrand $L(t_*, u_*(t_*), u'_*(t_*))$ as a Taylor series in t, keeping u_* fixed. We get

$$L(t_*, u_*(t_*), u'_*(t_*)) = L(t, u_*(t)u'_*(t)) + (t_* - t)\frac{d}{dt}L$$

but $(t_* - t) = \epsilon\xi$. We write

$$L(t_*, u_*) + L(t, u_*) + \epsilon\xi\frac{d}{dt}L$$

Similarly, we can expand

$$L(t, u_*) = L(t, u) + (u_* - u)\frac{d}{du}L$$

We now use this fact in the identity

$$\int_a^b L(t, u, u')\ dt = \int_a^b L(t_*, u_*, u'_*)\ dt_*$$

We know that $t_* = t + \epsilon\xi$

$$\Rightarrow \quad dt_* = dt + \epsilon\xi_t dt = (1 + \epsilon\xi)dt$$

so the right hand side is

$$\int_a^b \left(L + \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u' + \epsilon\xi\frac{d}{t}L\right)(1 + \epsilon\xi_t)dt$$

We may ignore terms that contain ϵ^2

$$\int_a^b L + \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u' + \epsilon\xi\frac{d}{dt}L + \epsilon\xi_t \, dt$$

Here the L will cancel the LHS and $\epsilon\xi\frac{d}{dt}L + \epsilon\xi_t$ is a perfect derivative; $(\epsilon\xi L)_t$ We thus get

$$\int_a^b \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u'\right) \, dt + [\epsilon\xi L]_a^b = 0$$

Integration by parts gives

$$\int \left(\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'}\right)\delta u \, dt + \left[\frac{\partial L}{\partial u'}\delta u + \epsilon\xi L\right]_a^b = 0$$

We have cheated a little bit here by using δu but it's fine.
The integral is zero by Euler-Lagrange equations. Thus, the expresion in brackets attains the same value at the endpoints a, b (which are arbitrary). Thus

$$\epsilon\xi L + \frac{\partial L}{\partial u'}\delta u = \text{constant}$$

We wish to show

$$\eta\frac{\partial L}{\partial u'} + \xi\left(L - u'\frac{\partial L}{\partial u'}\right)$$

is constant, namely

$$\xi L + \frac{\partial L}{\partial u'}(\eta - \xi u') = \text{constant}$$

we have to show that

$$\delta u = \epsilon\eta = \epsilon\xi u'$$

In fact

$$u_*(t_*) - u(t) = \epsilon\eta$$

and we need an expression for

$$\delta u = u_*(t) - u(t)$$

$$u_*(t) - u_*(t_*) + u_*(t_*) - u(t)$$

but $u_*(t) - u_*(t_*) = (t - t_*)\frac{du}{dt}$ and $u_*(t_*) - u(t) = \epsilon\eta$, so

$$\delta u = -\epsilon\xi u' + \epsilon\eta$$

□

This proof is not on the exam!

LECTURE 26: 30th March 10am

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8 Final Exam topics

1. Characteristics
2. Boundary value Problems
3. Eigenvalue Problems
4. Variational problems

One problem per topic a 3/4 exam. You can just forget about one topic if you want, thus you can pass on just 6 lectures!

8.1 To Know

- formulas for Fourier series (coefficients)
- charasteric equations (for $au_x + bu_y = c$ for for fully non-linear (use or derive)
- Euler-Lagrange equations $\frac{\partial L}{\partial u} = \frac{d}{dt}\frac{\partial L}{\partial u'}$
- Don't need to memorize other equations
- various proofs except for Banach/Alaoglu, Rellich's + Noether's

8.2 Sample question for 3

1. Show that the eigenvalues for $-\nabla u = \lambda u$ are positive when $u = 0$ on ∂A (will be more precise on the exam)
2. Show that the smallest eigenvalue corresponds to the minimum of the Rayleigh quotient. You may assume the minimum is attained.
3. Find eigenvalues/eigenfunctions for $-u'' + 3u' = \lambda u$ with $u(0) = u(2) = 0$