Chern-Simons Theory

Consider Chern-Simons Theory[1] with action

$$S_{CS} = \int \mathrm{d}^3x \left\{\mathcal{L}_{CS}\right\} \tag{1}$$

where the Lagrangian $\mathcal{L}_{CS}$ is given by

$$\mathcal{L}_{CS} = \frac{1}{2}\kappa\varepsilon^{\mu\nu\rho}A_\mu(x)\partial_\nu A_\rho(x) - A_\mu(x)J^\mu(x) \tag{2}$$

where $A_\mu(x)$ is a massless 1-form Abelian gauge field and $J^\mu(x)$ is a source term which vanishes at large distance.

(a) Show that the action (1) is invariant under U(1) gauge transformations of the form

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \partial_\mu\xi(x) \tag{3}$$

where $\xi(x)$ is some harmonic scalar field which vanishes at large distances.

(b) Find the homogeneous[2] field equations for $A_\mu(x)$ and show that they are gauge-equivalent to a trivial solution.

(c) Find the inhomogeneous[3] [4] field equations for $A_\mu(x)$ with

$$J^\mu(x) = \left(\rho, \vec{J}\right)$$

Consider now Maxwell-Chern-Simons Theory with source-free[5] Lagrangian

$$\mathcal{L}_{MCS} = -\frac{1}{4e^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\kappa\varepsilon^{\mu\nu\rho}A_\mu(x)\partial_\nu A_\rho(x) \tag{4}$$

[1] Note that Chern-Simons theory is only defined in (2+1)-dimensions. Compare this to the case of Maxwell theory which is defined in d-dimensions.

[2] That is,

$$J^\mu(x) = 0$$

[3] Note that in (2+1)-dimensions the electric and magnetic fields E and B are given by

$$B = \frac{1}{2}\varepsilon^{ij}\mathcal{F}_{ij} \quad \text{and} \quad E^i = -\mathcal{F}^{0i}$$

respectively, where

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

is the usual electromagnetic field-strength tensor.

[4] In addition, note that in (2+1)-dimensions the magnetic field B is a psuedoscalar.

[5] That is,

$$J^\mu(x) = 0$$

(d) Find the field equations for $A_\mu(x)$.

(e) Find the Hamiltonian for the system in the Coulomb gauge where

$$\vec{\nabla} \cdot \vec{A}(x) = 0$$

with

$$A_0(x) = 0$$