25/03/2015 MA 1214 **Group Theory - Homework 9**

1. (i) - We wish to show that given groups G_1 and G_2 then

$$G_1 \times \{e\} \trianglelefteq G_1 \times G_2$$

~ Well, clearly $G_1 \times \{e\}$ is a group

~ Thus it remains to show that for each $(g, e) \in G_1 \times \{e\}$ we get

$$(g_1, g_2) \circ (g, e) \circ (g_1^{-1}, g_2^{-1}) \in G_1 \times \{e\}$$

for all $(g_1, g_2) \in G_1 \times G_2$

~ But indeed

$$\begin{aligned}(g_1, g_2) \circ (g, e) \circ (g_1^{-1}, g_2^{-1}) &= (g_1 \circ g \circ g_1^{-1}, g_2 \circ e \circ g_2^{-1}) \\ &= (g_1 \circ g \circ g_1^{-1}, e)\end{aligned}$$

with $(g_1 \circ g \circ g_1^{-1}, e) \in G_1 \times \{e\}$

(ii) - We wish to show that

$$(G_1 \times G_2)/(G_1 \times \{e\}) \cong G_2$$

~ Well, let

$$\begin{aligned}\rho : (G_1 \times G_2)/(G_1 \times \{e\}) &\longrightarrow G_2 \\ (e, g) &\longmapsto \rho(e, g) = g\end{aligned}$$

~ Then ρ is an isomorphisim since

$$\begin{aligned}\rho((e, g_1) \circ (e, g_2)) &= \rho(e, g_1 \circ g_2) \\ &= g_1 \circ g_2 \\ &= \rho(e, g_1) \circ \rho(e, g_2)\end{aligned}$$

and ρ is injective since if

$$\rho(e, g_1) = \rho(e, g_2)$$

then we get
$$f_1 = f_2$$
and finally ρ is surjective since
$$Im(\rho) = G_2$$
■

(iii) - We wish to prove that the intersection of any number of normal subgroups of a group G is a normal subgroup of G

~ Let $N_1 \triangleleft G$, $N_2 \triangleleft G$, ..., $N_k \triangleleft G$ be normal subgroups

~ Then $\cap_{s\in S} N_s$ where $S \subseteq I$ with
$$I = \{1, 2, ..., k\}$$
an index set is indeed a group since $e \in \cap_{s\in S} N_s$ where $e \in G$ is the identity element, and if $h \in N_{i_1}$, $h \in N_{i_2}$, ..., $h \in N_{i_m}$ such that $h \in N_{i_1} \cap N_{i_2} \cap \dots \cap N_{i_m}$ then $h^{-1} \in N_{i_1}$, $h^{-1} \in N_{i_2}$, ..., $h^{-1} \in N_{i_m}$ since N_{i_j} is a subgroup, whence $h^{-1} \in N_{i_1} \cap N_{i_2} \cap \dots \cap N_{i_m}$

~ Moreover, for each $h_i \in N_i$ we have $g h_i g^{-1} \in N_i$ for all $g \in G$ for each
$$i = 1, 2, ..., k$$

~ Hence for each $h \in \cap_{s\in S} N_s$ we see that $g h g^{-1} \in N_s$ for all $s \in S$ or $g h g^{-1} \in \cap_{s\in S} N_s$ for all $g \in G$, as required
■

25/03/2015 MA 1214 Group Theory - Homework 9

2. (i) - We wish to show that

$$\rho: \mathbb{Z}\times(\mathbb{Z}_2\times\mathbb{Z}_4)\longrightarrow\mathbb{Z}_2\times\mathbb{Z}_4$$
$$(n,([a],[b]))\longmapsto\rho(n,([a],[b]))=([n+a],[n+b])$$

is a homomorphism

- Well,

$$\rho(n+m,([a],[b])+([c],[d]))=\rho(n+m,([a+c],[b+d]))$$
$$=([n+m+a+c],[n+m+b+d])$$
$$=([n+a],[n+b])+([m+c],[m+d])$$
$$=\rho(n,([a],[b]))+\rho(m,([c],[d]))$$

(ii) - We wish to find the orbits of $\mathbb{Z}_2\times\mathbb{Z}_4$ under $\mathbb{Z}$ and the stabilizers of $\mathbb{Z}_2\times\mathbb{Z}_4$ under the action of $\mathbb{Z}$

- Well,

$$\mathbb{Z}_2\times\mathbb{Z}_4=\{([0],[0]),([0],[1]),([0],[2]),([0],[3]),([1],[0]),([1],[1]),([1],[2]),([1],[3])\}$$

- Then

$$\mathbb{Z}([0],[0])=\{([0],[0]),([1],[1]),([0],[2]),([1],[3])\}$$
$$=\mathbb{Z}([1],[1])$$
$$=\mathbb{Z}([2],[2])$$
$$=\mathbb{Z}([3],[3])$$

while

$$\mathbb{Z}([1],[0])=\{([1],[0]),([0],[1]),([1],[2]),([0],[3])\}$$
$$=\mathbb{Z}([0],[1])$$
$$=\mathbb{Z}([1],[2])$$
$$=\mathbb{Z}([0],[3])$$

~Furthermore

$$\mathbb{Z}_{([a],[b])} = 4\mathbb{Z}$$

for each $([a],[b]) \in \mathbb{Z}_2 \times \mathbb{Z}_4$

©David-Alexander Robinson

©David-Alexander Robinson

25/03/2015 MA 1214 Group Theory - Homework 9

3. (i) - We wish to find the order of $\mathbb{Z}_8/\langle[2]_8\rangle$ and to check if it is cyclic

~ Well,

$$|\mathbb{Z}_8| = 8$$

with

$$\mathbb{Z}_8 = \{[0],[1],[2],[3],[4],[5],[6],[7]\}$$

while

$$\langle[2]_8\rangle = \{[2]_8,[4]_8,[6]_8,[0]_8\}$$

such that

$$|\langle[2]_8\rangle| = 4$$

~ Thus

$$|\mathbb{Z}_8/\langle[2]_8\rangle| = 8/4$$
$$= 2$$

~ Namely,

$$\mathbb{Z}_8/\langle[2]_8\rangle = \{\{[0],[2],[4],[6]\},\{[1],[3],[5],[7]\}\}$$
$$\cong \mathbb{Z}_2$$

~ Hence $\mathbb{Z}_8/\langle[2]_8\rangle$ is cyclic with

$$\mathbb{Z}_8/\langle[2]_8\rangle = \langle\{[1],[3],[5],[7]\}\rangle$$

(ii) - We wish to find the order of $\mathbb{Z}_8/\langle[3]_8\rangle$ and to check if it is cyclic

~ Well

$$\langle[3]_8\rangle = \{[3]_8,[6]_8,[9]_8,[12]_8,[15]_8,[18]_8,[21]_8,[24]_8\}$$
$$= \{[3]_8,[6]_8,[1]_8,[4]_8,[7]_8,[2]_8,[5]_8,[0]_8\}$$

or
$$\langle [3]_8 \rangle \cong \mathbb{Z}_8$$
whence
$$|\langle [3]_8 \rangle| = 8$$

~ Hence
$$|\mathbb{Z}_8 / \langle [3]_8 \rangle| = 8/8$$
$$= 1$$

~ Namely
$$\mathbb{Z}_8 / \langle [3]_8 \rangle \cong \mathbb{Z}_8 / \mathbb{Z}_8$$
$$= \{e\}$$

~ Thus $\mathbb{Z}_8 / \langle [3]_8 \rangle$ is cyclic with
$$\mathbb{Z}_8 / \langle [3]_8 \rangle = \langle [0]_8 \rangle$$