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1. (I) - We wish to check if

$$(x_1, y_1) \sim (x_2, y_2)$$

if

$$x_1 = x_2$$

is an equivalence relation

- Well,

$$x_1 = x_1$$

and if

$$x_1 = x_2$$

then

$$x_2 = x_1$$

and finally if

$$x_1 = x_2 \quad \text{and} \quad x_2 = x_3$$

then

$$x_1 = x_3$$

and so this is an equivalence relation with equivalence classes

$$[(x,y)] = \{(a,b) \in \mathbb{R}^2 : a = x\}$$

(II) - We wish to check if

$$(x_1, y_1) \sim (x_2, y_2)$$

if either

$$x_1 = x_2 \quad \text{or} \quad y_1 = y_2$$

is an equivalence relation

- Well, if

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{and} \quad (x_2, y_2) \sim (x_3, y_3)$$

with

$$x_1 = x_2 \quad \text{and} \quad y_2 = y_3$$

then we do not have

$$(x_1, y_1) \sim (x_3, y_3)$$

~Thus this is not an equivalence relation

(III) -We wish to check if

$$(x_1, y_1) \sim (x_2, y_2)$$

if $(y_1 - y_2) \in \mathbb{Z}$

~Well,

$$y_1 - y_1 = 0$$

with $0 \in \mathbb{Z}$ and so

$$(x_1, y_1) \sim (x_1, y_1)$$

~Moreover, if

$$(x_1, y_1) \sim (x_2, y_2)$$

such that

$$y_1 - y_2 = n$$

say satisfies $n \in \mathbb{Z}$ then $-n \in \mathbb{Z}$ and so

$$(x_2, y_2) \sim (x_1, y_1)$$

~Finally, if

$$(x_1, y_1) \sim (x_2, y_2) \text{ and } (x_2, y_2) \sim (x_3, y_3)$$

such that

$$y_1 - y_2 = n \quad \text{and} \quad y_2 - y_3 = m$$

say satisfy $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ then

$$\begin{aligned} y_1 - y_3 &= y_1 - y_2 + y_2 - y_3 \\ &= n - m \\ &= p \end{aligned}$$

say with $p \in \mathbb{Z}$ and so

$$(x_1, y_1) \sim (x_3, y_3)$$

~Hence this is an equivalence relation with

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1. equivalence classes

$$[(x,y)] = \{(a,b) \in \mathbb{R}^2 : (y-b) \in \mathbb{Z}\}$$

2. (i) - We wish to prove that if

$$a | b \quad \text{and} \quad b | c$$

then

$$a | c$$

~ Well, if

$$a | b \quad \text{and} \quad b | c$$

then

$$b = na \quad \text{and} \quad c = mb$$

for some $n \in \mathbb{Z}$ and some $m \in \mathbb{Z}$, whence

$$c = m(na)$$
$$= pa$$

say

~ Hence

$$a | c$$

$\blacksquare$

(ii) - We wish to prove that if

$$a | b \quad \text{and} \quad b | a$$

then

$$a = \pm b$$

~ Well, if

$$a | b \quad \text{and} \quad b | a$$

then

$$b = na \quad \text{and} \quad a = mb$$

for some $n \in \mathbb{Z}$ and some $m \in \mathbb{Z}$, whence

$$a = m(na)$$

and so

$$mn = 1$$

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2. ~ Thus we must have either

$$m = 1 \text{ and } n = 1 \quad \text{or} \quad m = -1 \text{ and } n = -1$$

yielding

$$a = \pm 6$$

$\blacksquare$

3. (I) - We wish to write (21, 5) and (-17, 5) in the form

$$a = nb + r$$

where $n \in \mathbb{Z}$ and $r \in \mathbb{N}$

- Then

$$21 = 4(5) + 1$$

while

$$-17 = -4(5) + 3$$

(II) - We wish to prove that if

$$m \mid n \quad \text{and} \quad a \equiv b \bmod n$$

then

$$a \equiv b \bmod m$$

- Well, if

$$m \mid n \quad \text{and} \quad a \equiv b \bmod n$$

then

$$n = pm \quad \text{and} \quad b = qn + a$$

for some $p \in \mathbb{Z}$, some $q \in \mathbb{Z}$

whence

$$b = q(pm) + a$$

and so

$$a \equiv b \bmod m$$

■

(III) - We wish to find those n such that

$$25 \equiv 1 \bmod n$$

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3. ~ Well,

$$25 \equiv 1 \mod n$$

if

$$25 = pn + 1$$

for some $p \in \mathbb{Z}$

~ That is, if

$$24 = pn$$

or, if

$$n | 24$$

~ Thus

$$n = 1, 2, 3, 4, 6, 8, 12, 24$$