28/01/2015 MA 1214 Group Theory - Homework 2

1. (i) - We wish to check if the binary operation $*$ given by

$$*: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$
$$(m,n) \mapsto m*n = mn$$

has an identity element

~ Well, for $1 \in \mathbb{Q}$ we get

$$\begin{aligned} 1*n &= 1.n \\ &= n \\ &= n*1 \end{aligned}$$

as required

(ii) - We wish to check if the binary operation

$$*: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$
$$(m,n) \mapsto m*n = m+n-1$$

has an identity element

~ Well, for $1 \in \mathbb{Q}$ we get

$$\begin{aligned} 1*n &= 1+n-1 \\ &= n \\ &= n*1 \end{aligned}$$

as required

(iii) - We wish to check if the binary operation

$$*: \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$
$$(m,n) \mapsto m*n = \tfrac{1}{2}(m+n)$$

has an identity element

~ But for each $m \in \mathbb{Q}$

$$m*m = \tfrac{1}{2}(m+m)$$

or

$$m * m = m$$

however this is not a unique identity element

(iv) - We wish to check if the binary operation

$$* : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$
$$(m, n) \mapsto m * n = -1$$

has an identity element

~ Well, for each $m \in \mathbb{Q}$ with

$$m \neq -1$$

there does not exist $n \in \mathbb{Q}$ such that

$$m * n = m$$

~ Hence there is no identity element

28/01/2015 MA 1214 Group Theory - Homework 2

2. - We wish to prove that

$$(ab)c = a(bc)$$

for any operation when either

$$a = e \quad \text{or} \quad b = e \quad \text{or} \quad c = e$$

where e is the identity element

~ Well, if

$$a = e$$

then

$$\begin{aligned}(ab)c &= [(e)b]c \\ &= [b]c \\ &= bc \\ &= e(bc) \\ &= a(bc)\end{aligned}$$

~ Similarly, if

$$b = e$$

then

$$\begin{aligned}(ab)c &= (ae)c \\ &= (a)c \\ &= ac \\ &= a(ec) \\ &= a(bc)\end{aligned}$$

~ Finally, if

$$c = e$$

then

$$\begin{aligned}(ab)c &= (ab)(e) \\ &= ab \\ &= a(be) \\ &= a(bc)\end{aligned}$$

3. (i) - We wish to check if (S, X) is a group where

$$S = \{-1, 1, 0\}$$

~ But there does not exist an inverse element $0^{-1} \in S$

~ Thus (S, X) is not a group

(ii) - We wish to check if $(S, +)$ is a group where

$$S = \{-1, 0, 1\}$$

~ But

$$1 + 1 = 2$$

with $2 \notin S$

~ Thus $(S, +)$ is not a group

(iii) - We wish to check if $(\mathbb{Z} \setminus \{0\}, X)$ is a group

~ But for each $n \in \mathbb{Z}$ with $n \neq 1$ and $n \neq -1$ there does not exist an inverse element $n^{-1} \in \mathbb{Z} \setminus \{0\}$

~ Thus $(\mathbb{Z} \setminus \{0\}, X)$ is not a group

28/01/2015 MA 1214 Group Theory - Homework 2

3. (iv) - We wish to check if $(5\mathbb{Z}, +)$ is a group

~ Well, we have an identity element $5 \in 5\mathbb{Z}$

~ Moreover, for each $n \in 5\mathbb{Z}$ we have an inverse element $n^{-1} \in 5\mathbb{Z}$ given by

$$n^{-1} = -n$$

~ In addition, addition is a closed operation such that

$$p = m + n$$

say, satisfies $p \in 5\mathbb{Z}$ for all $m, n \in 5\mathbb{Z}$

~ Finally, addition is an associative operation such that

$$(m+n)+p = m+(n+p)$$

for all $m, n, p \in 5\mathbb{Z}$

~ Hence $(5\mathbb{Z}, +)$ is a group

(v) - We wish to check if $(5\mathbb{Z}, \times)$ is a group

~ But there does not exist an inverse element $n^{-1} \in 5\mathbb{Z}$ for all

$$n \neq 5 \quad \text{and} \quad n \neq -5$$

~ Thus $(5\mathbb{Z}, \times)$ is not a group

(vi) - We wish to check if $(\mathbb{Z}, -)$ is a group

~ But subtraction is not an associative operation such that

$$(m-n)-p \neq m-(n-p)$$

for $m, n, p \in \mathbb{Z}$

~ Thus $(\mathbb{Z}, -)$ is not a group

(vii) - We wish to check if (S, X) is a group where

$$S = \{(-2)^n : n \in \mathbb{Z}\}$$

~ Well, we have an identity element $1 \in S$ where

$$(-2)^0 = 1$$

~ Moreover, for each $(-2)^n \in S$ we have an inverse element $[(-2)^n]^{-1} \in S$ given by

$$[(-2)^n]^{-1} = (-2)^{-n}$$

~ In addition, multiplication is an associative operation such that

$$[(-2)^n (-2)^m](-2)^p = (-2)^n[(-2)^m(-2)^p]$$

for all $(-2)^n, (-2)^m, (-2)^p \in S$

~ Finally, S is closed under multiplication since

$$(-2)^n(-2)^m = (-2)^p$$

say satisfies $(-2)^p \in S$ for all $(-2)^n, (-2)^m \in S$

~ Hence (S, X) is a group

28/01/2015 MA1214 Group Theory - Homework 2

4. (i) - We wish to prove that

$$(xy)^{-1} = y^{-1}x^{-1}$$

for all $x, y \in G$ where G is a group

~ Well, by definition

$$(xy)^{-1}(xy) = e$$

where $e \in G$ is the identity element, and so

$$\begin{aligned} ey^{-1}x^{-1} &= (xy)^{-1}(xy)y^{-1}x^{-1} \\ &= (xy)^{-1}x(e)x^{-1} \\ &= (xy)^{-1}(e) \end{aligned}$$

or

$$(xy)^{-1} = y^{-1}x^{-1}$$

■

(ii) - We wish to prove that

$$x^2 = x$$

if and only if

$$x = e$$

where $x \in G$ with G some group and $e \in G$ is the identity element

~ Well, if

$$x = e$$

then indeed

$$\begin{aligned} x^2 &= e^2 \\ &= e \\ &= x \end{aligned}$$

~ If on the other hand

$$x^2 = x$$

then

$$x^2x^{-1} = xx^{-1}$$
$$= e$$

or

$$x = e$$

■

(iii) - We wish to prove that

$$x^3 = x$$

does not have the unique solution

$$x = e$$

where $x \in G$ with G some group and $e \in G$ is the identity element

- Well, for

$$x = x^{-1}$$

we see that indeed

$$x^3 = x^2x^{-1}$$
$$= x$$

with

$$x \neq e$$

such as for

$$G = (\mathbb{Z}_2, +)$$

since

$$[1] + [1] + [1] = [3]$$
$$= [1]$$

with $[0] \in \mathbb{Z}_2$ the identity element