Mathematics for Students of Finance

An Introductory Course

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Abstract

Mathematics is an important tool for any student of Finance. In this note we review a selection of material typically covered by Undergraduate students of Engineering and Science. Special attention is paid to topics which the authors feel is of relevance to students of Finance. Those topics covered include introductory Real Analysis, introductory Linear Algebra, introductory material on functions of several real variables, and an introduction to differential equations.

This note is casual in style, and is accompanied by an extensive set of exercises prepared by Ms. Louise Spellacy[] for a previous version of this course.

Acknowledgements

Ms. Louise Spellacy, who wrote the original draft of these notes, which I have shameless used as a template. All mistakes are my own.

Dr. Mike Peardon[†] for coordinating the course and guiding us in the design and implimentation of the syllabus.

[†]

Preface

In what follows, a system of staring is used to indicate the relative difficulty and importance of the various topics discussed.

At least one of the authors feels that it would be worthwhile for students to become familiar with Mathematica, a copy of which is installed on all College computers, and can be obtained by students for use on personal laptops etc. through the IT Services webpages[3]

** Corresponds to material which will not be covered in full on the first reading

*** Corresponds to material which should be understood by any student of Finance but which regretably will not be covered due to time constraints

**** Corresponds to advanced material which will not be covered, but is included as a reference for further reading for students wishing to expand on the material covered. Most of this material is studied by students of an Undergraduate Physics or Engineering Degree

***** Corresponds to material which is not typically learnt by anyone expect Pure Math students, but which the author in his biased opinion feels is useful to know for those students wishing to persue quantative trading who have the time and motivation to learn it

Recommended Reading

- Calculus, Late Transcendentals, Anto, Bivens and Davis, 10th Ed. (2013)
- Elementary Linear Algebra, Anton and Rorres, 10th Ed. (2011)

[3] See IT Services - Student Software Information http://www.tcd.ie/itservices/software/kb/student_software.php

1 Functions

1.1 Numbers & Number Systems

$\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of counting numbers called the *Natural Numbers*

$\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ is the set of whole numbers, called *Integers*

$\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}\}$[4] is the set of fractions, called *Rational Numbers*

A *Radical* is a root of a rational number, such as $\sqrt{2}$ or $\sqrt[3]{5}$.

$\mathbb{R}$ is the set *Real Numbers*, it includes all rational numbers and all irrational numbers, such as radicals of positive rational numbers like $\sqrt{2}$, as well as special numbers like π and e [5] .

$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ where $i = \sqrt{-1}$ is the set of *Complex Numbers*, it includes all real numbers, as well as radicals of negative real numbers.

1.1.1 Indices

We recall the rules for *Indices*

$$x^0 = 1 \quad \text{and} \quad x^{-a} = \frac{1}{x^a}$$

with

$$x^a \cdot x^b = x^{a+b} \quad \text{and} \quad (xy)^a = x^a y^a \quad \text{and} \quad (x^a)^b = x^{ab}$$

We also note the relationship between indices and radicals, $\sqrt{x} = x^{1/2}$, or in general

$$\sqrt[n]{x^m} = x^{m/n}$$

1.2 Basic Properties of Functions

A *Function* f is a rule that assigns a unique real value $f(x)$ to each real number x.

We highlight the requirement that for a function f to be well defined it must assign a single number $f(x)$ to the number x.

Functions f and g may be added, subtracted, multiplied and divided. Thus

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (f \cdot g)(x) = f(x)g(x)$$

[4] Here the symbol $\in$, pronounced *'is an element of'*, denotes membership of a set.

[5] These special numbers are called *Transcendental Numbers.* Their rôle in mathematics is an interesting and imporant one, but beyond the scope of this course.

while

$$(f/g)(x) = f(x)/g(x)$$

provided that $g(x) \neq 0$.

1.2.1 Domain of a Function

The *Domain* of a function f is the set of allowable arguments that f can take

$$\text{Dom}(f) = \{x : f(x) \text{ is well defined } \}$$

In particular, we do not allow f to take an argument a which would give a nonesese result such as $f(a) = 1/0$. If such an a exists then $a \notin \text{Dom}(f)$[6] .

If f and g are two functions then[7]

$$\text{Dom}(f + g) = \text{Dom}(f) \cap \text{Dom}(g)$$

and $\text{Dom}(f \cdot g) = \text{Dom}(f) \cap \text{Dom}(g)$, while $\text{Dom}(f/g) = \text{Dom}(f) \cap \text{Dom}(g) \cap \{x : g(x) \neq 0\}$.

1.2.2 Range of a Function

The *Range* of a function f is the set results that f can give.

$$\text{Range}(f) = \{f(x) : x \in \text{Dom}(f)\}$$

1.2.3 Inverse of a Function

The *Inverse* of a function f, if it exists, is a function f^{-1} which has the property that

$$f^{-1}(f(x)) = x$$

That is, f^{-1} *'undoes'* whatever f did to x.

Note that $\text{Dom}(f) = \text{Range}(f^{-1})$.

[6] The symbol $\notin$ is pronounced *'is not an element of'*.

[7] The symbol $\cap$, pronounced *'intersection'*, means those elements which belong to both sets

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

On the other hand, the symbol $\cup$, pronounced *'union'*, means those elements which belong to either of the sets

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

1.2.4 Composition of Functions

There is another operation that may be performed on functions.

Composition is an operation on functions f and g for which $\mathrm{Dom}(f) = \mathrm{Range}(g)$ given by

$$(f \circ g)(x) = f(g(x))$$

That is, perform g first, and then take f of the result.

Exercises

[**?**] §1.1, Exercises 1 to 10 and 19 to 21.

1.2.5 Even and Odd Functions

We say that a function f is an *Even Function* if

$$f(-x) = f(x)$$

.

On the other hand, we say that f is an *Odd Function* if

$$f(-x) = -f(x)$$

Exercises

Exercise 1 Check if any of the following functions are even or odd or both

$$f_1(x) = \sin(x) \qquad g_1(x) = \cos(x) \qquad h_1(x) = 0$$
$$f_2(x) = x^2 \qquad g_2(x) = x^3 \qquad h_2(x) = 4x + 7$$

1.2.6 Polynomial Functions

A *Polynomial Function* f is a function wich can be written as a sum of powers of x

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

The a_i are called the *Coefficients*[8] of f.

[8] It is typical to set $a_n = 1$, which is achieved by dividing f by a_n.

The *Degree of a Polynomial Function* f is the highest power of x with nonzero coefficient. Thus

$$\deg(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = n$$

Similar to rational numbers $\mathbb{Q}$, we can have rational polynomials

$$f(x) = \frac{g(x)}{h(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}{b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m}$$

Also similar to rational numbers, we can perform *Long Division of Polynomial Functions* with remainders.

Exercises

Exercise 1 Simplify each of the following, by using long division of polynomials

$$f_1(x) = \frac{2x^3 + 7x^2 - 9}{x + 3} \qquad f_2(x) = \frac{x^4 + 11x^3 + 21x^2 - 59x - 70}{x - 2}$$

$$g(x) = \frac{2x^4 - 17x^3 - 25x^2 + 129x + 135}{x^3 - 11x^2 + 15x + 27} \qquad h(x) = \frac{x^4 + 3x^3 - 3x^2 - 11x - 6}{x^2 + 4x + 3}$$

1.2.7 Roots of Functions

A *Root* of a function f is a point $a \in \text{Dom}(f)$ such that $f(a) = 0$.

A polynomial function of degree d will have d roots over $\mathbb{C}$, counting multiplicity[9] .

If a is a root of the polynomial f then $(x - a)$ divides into f with zero remainder such that

$$f(x) = (x - a) \cdot g(x) \qquad \text{if } f(a) = 0$$

for some polynomial g with $\deg(g) < \deg(f)$.

For a polynomial of degree two[10], (called a *Quadratic Equation*) the roots are given by the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

[9] That is, if a root repeats it is counted however many times it repeats.

[10] Similar expressions exist for the roots of cubic equations (polynomials of degree three) and quartic equations (polynomials of degree four), but in practice they are too difficult for everyday use. Famously, no such general expression exists for quintic equations (polynomials of degree five) or higher order polynomials, as was proved by Able and Ruffini. The reason for this is explained by the the fascinating branch of mathematics known as *Galois Theory*.

1.2.8 Graph of a Function

Exercises

[?] §1.1, Exercises 1 to 13 and 15 to 17.

[?] §1.3, Exercises 1 to 9.

1.3 Limits and Continuity

1.3.1 Limit of a Function

The *Limit of a Function* f at a point a, if it exists, is the number L if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

A function can have at most one limit at any given point. In particular, if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$ then we must have that

$$L = M$$

We note that *'the sum of the limits is the limit of the sum'*, and similarly for products and recripicals. That is, if $\lim_{x\to a} f(x) = L$ and$\lim_{x\to a} g(x) = M$ then

$$\lim_{x\to a}(f + g)(x) = L + M \quad \text{and} \quad \lim_{x\to a}(f \cdot g)(x) = L \cdot M$$

and if $L \neq 0$ then

$$\lim_{x\to a} \left({}^1\!/\!_f\right)(x) = {}^1\!/\!_L$$

Exercises

[?] §1.2, Exercises 1 to 14.

1.3.2 Continuous Functions

We say that a function f is *Continuous* at the point a if

$$\lim_{x\to a} f(x) = f(a)$$

Intuitively, a function is continus if its graph can be drawn *'without needing to take the pen up off the page'*.

Similarly to limits, being continuous *'adds'* in the expected way. That is, if f and g are continuous at a, then $f + g$ is continuous at a, $f \cdot g$ is continuous at a, and provided that $f(a) \neq 0$ then '$nicefrac1f$ is continuous at a.

In addition, if g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ is continuous at a.

If a function f is continuous at x for each x in some interval (a, b)[11] then we say that f is *Continuous* on the interval (a, b).

Exercises

Exercise 1 Find the values of k such that the function f is continuous.

$$f(x) = \begin{cases} {}^{3\cos(2\pi x)}/_{x} & \text{if } x \neq 1 \\ k(k+2) & \text{if } x = 1 \end{cases}$$

Exercise 2 Show that the function $f(x) = |x - 2|$ is continuous.

Exercise 3 Show that the limit $\lim_{x\to 0} {}^{|x|}/_{x}$ does not exist.

Exercise 4 Given $\varepsilon > 0$, find a $\delta > 0$ to prove that

$$\lim_{x\to 3} 5x - 2 = 13$$

Exercise 5 Given $\varepsilon > 0$, find a $\delta > 0$ to prove that

$$\lim_{x\to -1/2} \frac{4x^2 - 1}{2x + 1} = -2$$

Exercise 6 Given $\varepsilon > 0$, find a $\delta > 0$ to prove that

$$\lim_{x\to 3} x^2 + x = 12$$

1.3.3 Asymptotes

A function f has a *Verticle Asymptote* at the line $x = a$ if[12]

$$\lim_{x\to a} f(x) = \pm\infty$$

On the other hand, f has a *Horizontal Asymptote* at the line $y = b$ if either

$$\lim_{x\to -\infty} f(x) = b \quad \text{or} \quad \lim_{x\to +\infty} f(x) = b$$

[11] Here $(a, b) = \{x : a < x < b\}$ is the *Open Interval* a-b. It does not include the points a and b. In contrast $[a, b] = \{x : a \leq x \geq b\}$ is the *Closed Interval* a-b, which includes the points a and b.

[12] Note that this limit need only be attained when coming from at least one direction.

Exercises

[?] §1.1, Exercise 14.

Exercise 1 Find the verticle asymptotes and horizontal asymptotes of the following rational functions, if and exist

$$f(x) = \frac{3x-6}{2x^2-2x-4} \qquad g(x) = \frac{6x^3+3x+1}{x^2-3x+2}$$

1.3.4 Theorems on Bounded Continuous functions**

In what follows, let f be a function and let a and b be points.

Theorem 1: If f is continuous on $[a, b]$ with $f(a) < 0 < f(b)$ then there is some $x \in [a, b]$ such that $f(x) = 0$.

Theorem 2: If f is continuous on $[a, b]$ then f is bounded from above on $[a, b]$.

That is, there is some $N \in \mathbb{R}$ such that $f(x) < N$ for all $x \in [a, b]$.

Theorem 3: If f is continuous on $[a, b]$ then there is some $y \in [a, b]$ such that $f(y) \geq f(x)$ for all $x \in [a, b]$.

This Theorem states that a continuous function on a closed interval takes its maximum value on said interval.

These theorems give us some *'global'* properties of a continuous function on an interval.

Theorem 4: If f is continuous on $[a, b]$ with $f(a) < c < f(b)$ then there is some $x \in [a, b]$ such that $f(x) = c$.

Note that this is simply a special case of the first Theorem.

Similarly

Theorem 5: If f is continuous on $[a, b]$ with $f(a) > c > f(b)$ then there is some $x \in [a, b]$ such that $f(x) = c$.

Together Theorems 4 and 5 are known as the *Intermediate Value Theorem*. They state that if a continuous function on an interval takes on two values, then it also takes on every value in between these two values.

Theorem 6: If f is continuous on $[a, b]$ then f is bounded from below on $[a, b]$.

That is, there is some $M \in \mathbb{R}$ such that $f(x) > M$ for all $x \in [a, b]$.

Together Theorems 2 and 6 state that a continuous function on a bounded inteval is bounded on that interval. In particular, $|f(x)|_{x \in [a,b]} < \max(N, M)$.

Theorem 7: If f is continuous on $[a, b]$ then there is some $y \in [a, b]$ such that $f(y) \leq f(x)$ for all $x \in [a, b]$.

That is,a continuous function on a closed interval takes its minimum value on said interval.

1.3.5 Least Upper Bounds & Greatest Lower Bounds****

1.3.6 L'Hôpital's Rule**

If $f(x) = {}^{g(x)}/_{h(x)}$ with $g(a), h(a) \in \{+\infty, -\infty, 0\}$ for some point a then *l'Hôpital's Rule* states that

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{g'(a)}{h'(a)}$$

Exercises

[?] §2.5, Exercises 1 to 8.

Exercise 1 Calculate each of the following limits[13]

$$(i) \lim_{x\to 3\pi} \frac{\sin(5x)}{x-3\pi} \qquad (ii) \lim_{x\to\infty} \frac{x^{257}}{\exp(257x)}$$

$$(iii) \lim_{x\to\infty} \frac{\log(x)}{x^{1/2}} \qquad (iv) \lim_{x\to\pi} \frac{\sin(7x)}{\tan(3x)}$$

$$(v) \lim_{x\to 0} \frac{\big(\log(x)\big)^7}{x} \qquad (vi) \lim_{x\to\infty} \frac{\big(\log(x)\big)^2}{x}$$

$$(vii) \lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\exp(x-1)-x}\right) \qquad (viii) \lim_{x\to\infty} \sinh(x)$$

$$(ix) \lim_{x\to 3} \sinh(x-x).\log(x-3) \qquad (x) \lim_{x\to 0} \sinh(x).\log(x)$$

1.4 Transcendental Functions**

1.4.1 Trigonometric Functions

The *Trigonometric Functions* sin and cos are the unique functions with the properties

$$\sin'(x) = \cos(x) \quad \text{and} \quad \cos'(x) = -\sin(x)$$

and

$$\sin(0) = 0 \quad \text{and} \quad \cos(0) = 1$$

and

$$|\sin(x)| < 1 \quad \text{and} \quad |\cos(x)| < 1$$

[13] Note that familiarity with transcendental functions and their derivatives will be needed, which are introduced in later sections. It is recommend to study those sections first before returning to these exercises

We may define the other trigonometric functions in terms of sin and cos by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

while

$$\sec(x) = \frac{1}{\cos(x)} \quad \text{and} \quad \operatorname{cosec}(x) = \frac{1}{\sin(x)} \quad \text{and} \quad \cot(x) = \frac{1}{\tan(x)}$$

1.4.2 Inverse Trigonometric Functions

The inverse functions of the trigonometric functions sin and cos are defined by the functions arcsin and arccos respectively[14]

$$\arcsin\big(\sin(x)\big) = x \quad \text{and} \quad \arccos\big(\cos(x)\big) = x$$

In a similar way, inverse functions arctan, arcsec, arccosec and arccot are defined.

1.4.3 The Logarithm Function

The *Logarithm Function* log is definted by the definite integral[15]

$$\log(x) = \int \frac{1}{x}\,\mathrm{d}x$$

It is the unique function with the properties that

$$\log(x \cdot y) = \log(x) + \log(y) \quad \text{and} \quad \log(1) = 0$$

1.4.4 The Exponential Function

The *Exponential Function* exp is defined as the inverse of the logarithm function such that

$$\exp\big(\log(x)\big) = x$$

It is the unique function with the properties that

$$\exp'(x) = \exp(x)$$

[14] Note that many sources (including this author) use the notation $\sin^{-1}$ and $\cos^{-1}$ to denote the inverse functions of sin and cos, however this is in general considered incorrect, since sin and cos are not injective, and thus strictly speaking they do not have inverse functions.

[15] The author is aware that we have yet to formally define the notions of differentiation and integration, but has decided to follow the so called *'Early Transcendentals'* approach to teaching Calculus in light of the short amount of time avaliable.

and

$$\exp(x+y) = \exp(x)\cdot\exp(y) \quad \text{and} \quad \exp(0) = 1$$

Exercises

[?] §1.5, Exercises 1 to 6.

1.4.5 Hyperbolic Trigonometric Functions**

As well as the regular trigonometric functions, we can define *Hyperbolic Trigonometric Functions*[16] $\sinh(x)$ and $\cosh(x)$[17]

$$\sinh(x) = \frac{1}{2}\big(\exp(x) - \exp(-x)\big) \qquad \text{and} \qquad \cosh(x) = \frac{1}{2}\big(\exp(x) + \exp(-x)\big)$$

In addition, we may define hyperbolic anologs of the other usual trigonometric functions[18]

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

while[19]

$$\operatorname{cosech}(x) = \frac{1}{\sinh(x)} \quad \text{and} \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)} \quad \text{and} \quad \coth(x) = \frac{1}{\tanh(x)}$$

These functions have trigonometric identities similar to the usual functions sin and cos, but with some subtle differences

$$\cosh^2(x) - \sinh^2(x) = 1$$

As in the case of the regular trigonometric functions, inverse functions may be defined for the hyperbolic trigonometric functions

$$\operatorname{arcsinh}\big(\sinh(x)\big) = x \quad \text{and} \quad \operatorname{arccosh}\big(\cosh(x)\big) = x$$

and moreover arctanh, arcsech, arccosech and arccoth are defined.

Exercises

[16] These Hyperbolic Trigonometric Functions play an important role in Complex Analysis, but unfortunately we not be able to explore this further, due to time constraints.

[17] Pronounced *'sin-ch'* and *'cos-sh'* respectively.

[18] Pronounced *'tan-ch'*.

[19] Pronounced *'co-sech'*, *'se-ch'* and *'co-th'* respectively.

2 Differeniation

2.1 Introduction to Differentiation

The *Derivative* of a continuous function f at a point a, if it is defined, is given by the limit

$$f(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

If the derivative of a function f exists at every point x in some interval (a, b) then we say that f is *Differentiable* on (a, b).

If the derivative f' is itself differentiable on (a, b), then we say that f is *Twice Differentiable* on (a, b).

Continuing in this way, if f is differentiable an infinite number of times on (a, b), then we say that f is a *Smooth* on (a, b).

For a polynomial function $f(x) = x^n$ we get

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$$

It follows that for a constant function $f(x) = c$ for all x then

$$\frac{\mathrm{d}}{\mathrm{d}x} c = 0$$

since $c = cx^0$.

Differentiation adds in the expected way

$$(f + g)'(a) = f'(a) + g'(a)$$

2.1.1 The Product Rule

However, differentiation does not multiply as expected. Instead by *The Product Rule*

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

2.1.2 The Quotient Rule

Simarily, for a quotent we get *The Quotient Rule*

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a) \cdot f'(a) - g'(a) \cdot f(a)}{\big(g(a)\big)^2}$$

2.1.3 The Chain Rule

Finally, for the composition of two functions we get *The Chain Rule*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Exercises

[?] §2.1, Exercises 5 to 9.

[?] §2.2, Exercises 2 and 4.

2.1.4 Derivatives of Trignometric Functions

As per the definition of the trigonometric functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$$

The derivatives of the other trigonometric functions tan, sec, cosec and cot may be derived using the above rules.

2.1.5 Derivatives of Inverse Trignometric Functions

As per the definition of the inverse trigonometric functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

2.1.6 Derivative of The Exponential Function

As per the definition of the exponential function

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x)$$

2.1.7 Derivative of The Logarithm Function

As per the definition of the logarithm function

$$\frac{\mathrm{d}}{\mathrm{d}x}\log(x) = \frac{1}{x}$$

2.1.8 Derivatives of Hyperbolic Trigonometric Functions**

As per the definition of the hyperbolic trigonometric functions in terms of exponential functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh(x) = \cosh(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \sinh(x)$$

The derivatives of the other hyperbolic trigonometric functions tanh, sech, cosech and coth may then be derived using the above rules.

Exercises

[?] §2.1, Exercises 12 and 13.

[?] §2.2, Exercises 1 and 5.

[?] §2.4, Exercises 1 to 4.

2.1.9 Implicit Differentiation

Exercises

[?] §2.3, Exercises 1, 3, 4, 6 and 6.

2.1.10 Rolle's Theorem**

2.1.11 The Mean Value Theorem**

2.2 Applications of Differentiation

We can use differentiation to find out useful information about a function, such as where it attains it's maximum value.

2.2.1 Tangent Lines and Slopes of Functions

The derivative of a function $f'(x)$ gives us an equation for the tangent to the function.

Moreover, at any point a, the derivative $f'(a)$ gives us the slope of the function at the point a.

Exercises

[?] §2.1, Exercises 1 to 3, and 11.

[?] §2.2, Exercises 3.

[?] §2.3, Exercises 2 and 5.

2.2.2 Critical Points

The point a is called a *Critical Point* of a function f if

$$f'(a) = 0$$

Critical points are typically the turning points of a function.

2.2.3 Local Minima and Local Maxima of a Function

The point a is called a *Local Minima* of the function f if

$$f''(a) > 0$$

On the other hand, a is called a *Local Maxima* if

$$f''(a) < 0$$

2.2.4 Inflexion Points

The point a is called an *Inflexion Point* of the function f if

$$f''(a) = 0$$

2.3 Increasing and Decreasing Functions

We say that a function f is *Increasing* at the point a if

$$f'(a) > 0$$

On the other hand, f is *Decreasing* at a if

$$f'(a) < 0$$

If f is increasing at x for every x in some interval (a, b) then we say that f is increasing on the interval (a, b).

Similarly, if f is decreasing at x for every $x \in (a, b)$ then we say that f is decreasing on (a, b)

Intuitively, the graph of the function will be going upwards (when reading from left-to-right) wherever it is increasing.

On the other hand, the graph of the function will be going downwards (when reading from left-to-right) wherever it is decreasing.

2.3.1 Concave-Up and Concave-Down Functions

We say that a function f is *Concave-Up*[20] at the point a if

$$f''(a) > 0$$

On the other hand, f is *Concave-Down*[21] at a if

$$f''(a) < 0$$

If f is concave-up at x for every x in some interval (a, b) then we say that f is concave-up on the interval (a, b).

Similarly, if f is concave-down at x for every $x \in (a, b)$ then we say that f is concave-down on (a, b).

Intuitively, the graph of the function will look like a *'smile'* where it is concave-up.

On the other hand, the graph of the function will look like a *'frown'* where it is concave-down.

2.3.2 The Derivative and Plotting Functions‡‡

All of this information together can help us to plot a given function.

Exercises

Exercise 1 Plot each of the following functions by finding any roots, any critical points, any points of inflexion, any verticle asymptotes, any horizontal asymptotes, any regions where the function is increasing/ decreasing, or any regions where the function is concave-up/concave-down

$$(i) f_1(x) = \qquad (ii) f_2(x) =$$
$$(iii) f_3 = \qquad (iv) f_4(x) =$$

[20] Note that several books use the term *Convex* or *Convex-Down* to mean what we call concave-up.
[21] Note that several books use the term *Convex-Up* to mean what we call concave-down.

3 Integration

3.1 Sequences & Series

A *Sequence* $(a_n)_{n=0}^{\infty}$ is a function whos domain is some subset of the natural numbers $\mathbb{N}$.

A *Series* $\sum_{n=0}^{\infty} a_n$ is a sum of the terms of a sequence $(a_n)_{n=0}^{\infty}$.

The *k-th Partial Sum* S_k of the sequence $(a_n)_{n=0}^{\infty}$ is the sum of the first k terms of the sequence

$$S_k = (a_n)_{n=0}^{k}$$

We say that the series $(a_n)_{n=0}^{\infty}$ is *Convergent* if the limit $\lim_{k\to\infty} S_k$ of the partial sums exists.

If the series does not converge then we say that the series *Diverges*.

3.1.1 Sums of Infinite Series

If $|r| < 1$ then

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$$

3.2 Tests for Convergence**

3.2.1 The Ratio Test

The *Ratio Test* states that the series $\sum_{n=1}^{\infty} a_n$ converges if

$$|\lim_{k\to\infty} {}^{a_{k+1}}/_{a_k}| < 1$$

If

$$|\lim_{k\to\infty} {}^{a_{k+1}}/_{a_k}| > 1$$

then the series diverges, while if

$$|\lim_{k\to\infty} {}^{a_{k+1}}/_{a_k}| = 1$$

then the test is inconclusive.

3.2.2 The Riemann zeta-Function & The Comparisson Test

The *Riemann ζ-Function* ζ is defined by[22]

$$\zeta(s) = \sum_{n=1}^{\infty}(1)(n^s)$$

The Riemann ζ-Function converges for $s > 1$.

For $s = 1$, the series is known as the *Harmonic Series*[23]

$$\zeta(1) = \sum_{n=1}^{\infty}(1)(n)$$
$$= 1 + frac12 + \frac{1}{3} + \dots$$

and is divergent.

The *Comparisson Test* states that the series $\sum_{n=1}^{\infty} a_n$ converges if there exists some $N \in \mathbb{N}$ and a convergent series $\sum_{n=1}^{\infty} b_n$ with $|a_n| < |b_n|$ for each $n > N$.

3.2.3 The Integral Test**

The *Integral Test* states that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} a(x)\,\mathrm{d}x$ converges.

3.2.4 Absolute Convergence

The series $\sum_{n=0}^{\infty} a_n$ is *Absolutely Convergent* if the series of absolute values $\sum_{n=0}^{\infty} |a_n|$ converges.

Theorem: An absolutely convergent series is convergent.

[22] Strictly speaking, we are studying the series as written down by Euler, since we restrict ourselves to the case of real variable s. The Riemann ζ-Function with complex variable plays an important and complex rôle in areas of maths ranging from Complex Analysis to advanced topics in Number Theory, but these are far beyond the scope of this course.

[23] This name is derived from the importance of the series in Music Thoery.

3.2.5 Uniform Convergence*****

3.3 The Taylor Series

If f is a smooth function then the *Taylor Series* of f about the point a is given by

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

where $f^{(n)}$ is the n-th derivative of f.

3.3.1 Taylor Series for the Exponential Function

The Taylor Series of the exponential function about the point 0 is given by

$$\begin{aligned} \exp(x) &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

3.3.2 Taylor Series for the Logarithm Function

The Taylor Series of the logarithm function about the point 0 is given by[24]

$$\begin{aligned} \log(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^n}{n} \end{aligned}$$

3.3.3 Taylor Series for the Trigonometric Functions

The Taylor Series of the trigonometric functions sin and cos about the point 0 are given by

$$\begin{aligned} \sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \cos(x) &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \end{aligned}$$

[24] Alternatively

$$\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^n}{n}$$

3.4 Integration

3.4.1 Riemann Sums**

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x_k \to 0} \sum_k f(x_k)\Delta x_k$$

3.4.2 Antiderivatives

We may perform the inverse of differentiation. Thus, if f is a continuous function, we *'guess'* a function F such that

$$F'(x) = f(x)$$

3.4.3 The Fundamental Theorem of Calculus**

If f is a continuous function on the interval $[a, b]$ then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

where $F'(x) = f(x)$.

3.4.4 Definite Integrals

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x_k \to 0} \sum_k f(x_k)\Delta x_k$$

3.4.5 Integration by Substitution

3.4.6 Integration by Parts

3.4.7 Integration by Partial Fractions

3.4.8 Improper Integrals

3.5 Area & Volume Integrals**

3.5.1 Arc Length

The *Arch Length* L_C of the curve $y = f(x)$ between the points a and b is given by

$$L_C = \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

3.5.2 Area Between Two Curves

The *Area* A between two curves $y_1 = f(x)$ and $y_2 = g(x)$ is given by

$$A = \int_a^b |f(x) - g(x)|\, \mathrm{d}x$$

where a and b are the points of intersection of y_1 and y_2.

3.5.3 Volume of Revolution Integrals

The *Volume of Revolution* V_x of the curve $y = f(x)$ between the points a and b about the x-axis[25] is given by

$$V_x = \pi \int_a^b f(x)^2\, \mathrm{d}x$$

The volume of revolution V between two curves $y_1 = f(x)$ and $y_2 = g(x)$ about the x-axis is given by

$$V = \pi \int_a^b |f(x)^2 - g(x)^2|\, \mathrm{d}x$$

where a and b are the points of intersection of y_1 and y_2.

3.5.4 Surface Area of Revolution Integrals

The *Surface Area of Revolution* A_x of the curve $y = f(x)$ between the points a and b about the x-axis is given by

$$A_x = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

Exercises

[25] Note that the volume of revolution about the x-axis V_x and the volume of revolution about the y-axis V_y are not in general the same, and care must be taken to calculate the correct value.

4 Differential Equations**

4.1 Differential Equations

4.2 Separable Differential Equations

4.3 Initial Conditions

4.4 First-Order Linear Ordinary Differential Equations

4.4.1 Inhomogenous Differential Equations

4.5 Numerical Methods

4.5.1 The Newton-Raphson Method

The *Newton-Raphson Method* allows us to calculate approximate roots of a differentiable function f, given an initial *'guess'* x_0[26], by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided that $f'(x_n) \neq 0$ for any n[27].

Exercises

[?] §2.14, Exercise 1, Exercises 2

Exercise 1 Use the Newton-Raphson Method to find an approximation to a root $x \in (0, +\infty)$ of the function

$$f(x) = x^5 - 5x^3 - 2$$

Exercise 2 Use the Newton-Raphson Method to find an approximation to a root $x \in (\pi/2, 3\pi/2)$ of the function

$$f(x) = 1 + x^2 \sin(x)$$

Exercise 3 Use the Newton-Raphson Method to find an approximation to the minimum of the function

$$f(x) = \frac{1}{4}x^4 + x^2 - 5x$$

[26] Recall that by Intermediate Value Theorem, if a function f is continuous on an interval (a, b) and $f(a)$ and $f(b)$ differ in sign, then f has at least one root in (a, b).

[27] It should be noted that the Newton-Raphson Method will not work in all cases due to the difficulties of numerical analysis, but the discussion of such is beyond the scope of this course.

Exercise 4 Use the Newton-Raphson Method to find an approximation to the maximum in the interval $[0, \pi]$ of the function

$$f(x) = x\sin(x)$$

4.5.2 The Euler Method

The *Euler Method* allows us to calculate approximate solutions to differential equations. We we restrict to the case of first order differential equations of the form

$$y'(x) = f(x, y) \quad \text{with initial condition} \quad y(x_0) = y_0$$

Then, by Euler,

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

where h is the *Stepsize*.

The error in the approximate solution y_{n+1} is proportional to h. Thus reducing the stepsize will reduce the error, however, doing so will increase the computational cost.

Exercises

Exercise 1 Use the Euler Method to sovle the following differential equation, using a step-size of $1/2$ for $0 \leq x \leq 4$

$$y'(x) = \sqrt[3]{y(x)} \quad \text{with initial condition} \quad y(0) = 1$$

Exercise 2 Use the Euler Method to sovle the following differential equation, using a step-size of $1/4$ for $0 \leq t \leq 2$

$$y'(x) = x - y(x)^2 \quad \text{with initial condition} \quad y(0) = 1$$

Exercise 3 Use the Euler Method to sovle the following differential equation, using a step-size of $1/10$ for $0 \leq t \leq 1$

$$y'(t) = \exp(-y(t)) \quad \text{with initial condition} \quad y(0) = 0$$

Exercise 4 Use the Euler Method to sovle the following differential equation, using a step-size of $1/10$ for $0 \leq t \leq 1$

$$y'(t) = \sin(\pi t) \quad \text{with initial condition} \quad y(0) = 0$$

4.5.3 The Runge-Kutta Method****

The *Runge-Kutta Method* is another method that allows us to calculate approximate solutions to differential equations of the form

$$y'(x) = f(x, y) \quad \text{with initial condition} \quad y(x_0) = y_0$$

In particular

$$y_{n+1} = y_n + 1/6\,(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = h\,f(x_n, y_n) \qquad k_2 = h\,f\,(x_n + 1/2h, y_n + 1/2k_1)$$

$$k_3 = h\,(x_n + 1/2h, y_n + 1/2k_2) \qquad k_4 = h\,f(x_n + h, y_n + k_3)$$

with h the *Stepsize*[28].

The error in the approximate solution y_{n+1} is proportional to h^4. Thus, for $h < 1$, the error in the Runge-Kutta method is far less than that of the Euler method.

However, the Runge-Kutta method is much more computationally expensive.

Again, reducing the stepsize will reduce the error further, however, doing so will increase the computational cost even more.

Exercises

Exercise 1 Use the Runge-Kutta Method to sovle the following differential equation, using a step-size of $1/4$ for $0 \leq x \leq 2$

$$y'(x) = -2y(x)^2 x \quad \text{with initial condition} \quad y(0) = 1$$

Exercise 2 Use the Runge-Kutta Method to sovle the following differential equation, using a step-size of $1/4$ for $-1 \leq x \leq 1$

$$y'(x) = \big(y(x) + x\big)^2 \quad \text{with initial condition} \quad y(-1) = -1$$

[28] Strictly speaking, this method is known as the *'Fourth Order* Runge-Kutta method. Other Runge-Kutta methods exist, but this is the classic one.

4.5.4 Numerical Solutions for Higher Order Differential Equations****

Exercises

Exercise 1 Use the Runge-Kutta Method to sovle the following third-order differential equation, using a step-size of $1/20$ for $0 \leq x \leq 50$[29]

$$y'''(x) + \alpha y'(x) - y'(x)^2 + y(x) = 0 \quad \text{with initial conditions} \quad y(0) = 1/50\,,\ y'(0) = 0\,,\ \text{ and } y''(0) = 0$$

where $\alpha = 2.017$.

Exercise 2 Use the Runge-Kutta Method to sovle the following third-order differential equation, using a step-size of $1/20$ for $0 \leq x \leq 50$

$$y'''(x) + \alpha y'(x) - y'(x)^2 + y(x) = 0 \quad \text{with initial conditions} \quad y(0) = 41/2000\,,\ y'(0) = 0\,,\ \text{ and } y''(0) = 0$$

where $\alpha = 2.017$.

[29] Note that this will require 1,000 steps, and so in no circumstance should it be done by hand. A simple script written in C++ or your favourite programing language should be able to find a solution in a number of seconds.

5 Calculus in Several Real Variables**

5.1 Functions of Several Real Variables

A *Functions of Several Real Variables* f is a rule that assigns a unique real number $f(x_1, x_2, \ldots, x_n)$ to each point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

In a similar manor to functions of a single real variable, we may define the notions of limits and continunity.

We can again prove that

- The composition of continuous functions is continuous
- The sum or product of continuous functions is continuous
- The quotient of continuous functions is continuous, so long as the denominator is not zero

In what follows, we will restrict our attention to functions of two variables $f(x, y)$ or three variables $f(x, y, z)$.

5.1.1 Partial Derivatives

If $f = f(x, y, z)$ is a function of several real variables, then we may calculate the derivative of f in the direction of each variable, called a partial derivative, denoted $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$[30] .

In particular, the *Partial Derivative* of f with respect to x at the point (x_0, y_0), if it exists, is the limit

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(x_0,y_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

Similar definitions exist for the partial derivatives in the y and z directions.

To calculate the partial derivative of f with respect to x, say, we treat all other variables y, z as constants and then use the usual rules of differentiation.

In addition, we may calculate higher order derivatives $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$ etc.

[30] Note that several books use a subscript notation to denote partial derivatives

$$f_x = \partial f/\partial x \text{ and } f_y = \partial f/\partial y \text{ etc.}$$

We will avoid this notation.

An important distinction now arrises for functions of several real variables, namely, we may calculate *Mixed Partial Derivatives*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Theorem: (Mixed Partials Commute)[31]. Let $f = f(x, y, z)$ be a function of several real variables with continuous second-order partial derivatives. Then the mixed partial derivatives of f commute.

That is

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

and similarly for other pairs of mixed second-order partial derivatives.

Exercises

Exercise 1 Calculate all first-order partial derivatives of the following functions of two real variables

$$f_1(x,y) = \log\left(x^2y^4 + \exp(3y)\right) \qquad g_1(x,y) = x\sqrt{y}\cos(x)$$
$$f_2(x,y) = \frac{yx^3}{4x+3y} \qquad g_2(x,y) = \cos(xy^2)$$

Exercise 2 Calculate all second-order partial derivatives of the following functions of two real variables, and confirm that the mixed partial derivatives commute

$$f_1(x,y) = 4x^2 - 8x^4y + 7y^4 - 3 \qquad g_1(x,y) = \exp(x - y^2)$$
$$f_2(x,y) = \frac{x-y}{x+y} \qquad g_2(x,y) = \log(x^2 + y^2)$$

Exercise 3 Calculate all third-order partial derivatives of the following functions of two real variables, and confirm that the mixed partial derivatives commute

$$f_1(x,y) = x^3y^5 - 2x^2y + x \qquad g_1(x,y) = \exp(y)\cos(x)$$

Exercise 4 Calculate all second-order partial derivatives of the following functions of three real variables, and confirm that the mixed partial derivatives commute

$$f_1(x,y,z) = x^3y^5z^7 + xy^2 + y^3z \qquad g_1(x,y,z) = (4x - 3y + 2z)^5$$

[31] This Theorem is attributed to many people, among them Schwarz, Clairaut, and Young.

Exercise 5 (The Laplace Equation). Show that the following functions $u(x,y)$ satisfy the *Laplace Equation*

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$$u_1(x,y) = x^2 - y^2 + 2xy \qquad u_2(x,y) = \log(x^2+y^2) + 2\arctan\left(\frac{y}{x}\right)$$

Exercise 6 (The Heat Equation). Show that the following functions $u(x,t)$ satisfy the *Heat Equation*

$$\frac{\partial u(x,t)}{\partial t} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

where c is some constant.

$$u_1(x,t) = \exp(-t)\sin\left(\frac{x}{c}\right) \qquad u_2(x,t) = \exp(-t)\cos\left(\frac{x}{c}\right)$$

Exercise 7 (The Cauchy-Riemann Equations). Show that the following pairs of functions $u(x,y)$ and $v(x,y)$ satisfy the *Cauchy-Riemann Equation*

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \quad \text{and} \quad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$$

$$u_1(x,y) = x^2 - y^2 \quad \text{and} \quad v_1(x,y) = 2xy$$
$$u_2(x,y) = \exp(x)\cos(y) \quad \text{and} \quad v_2(x,y) = \exp(x)\sin(y)$$

5.1.2 The Hessian

We may find critical points and local minima and maxima of a function $f = f(x,y,z)$ of several variables, similar to the case of function of a single real variable $g = g(x)$.

In waht follows we restrict our attention to functions of two variables[32].

In particular, a *Critical Point* of a function $f = f(x,y)$ is a point (x_0, y_0) such that the first order partial derivates of f vanish. That is,

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(x_0,y_0)} = 0 \quad \text{and} \quad \left.\frac{\partial f(x,y)}{\partial y}\right|_{(x,y)=(x_0,y_0)} = 0$$

[32] The following analysis can indeed be extended to functions of more than two variables, but becomes computationally more involved. In short, one must determine if the Hessian is positive-definite to indicate a local minimum, or negative-definite to indicate a local maximum.

The *Hessian* $H(x, y)$ is the matrix of second-order partial derivatives of f. That is,

$$H(x,y) = \begin{pmatrix} \frac{\partial^2 f(x,y)}{\partial x^2} & \frac{\partial^2 f(x,y)}{\partial x \partial y} \\ \frac{\partial^2 f(x,y)}{\partial y \partial x} & \frac{\partial^2 f(x,y)}{\partial y^2} \end{pmatrix}$$

If det $\big(H(x_0, y_0)\big) > 0$ and $\partial^2 f/\partial x^2|_{(x,y)=(x_0,y_0)} < 0$ then (x_0, y_0) is a *Local Maximum.*

If det $\big(H(x_0, y_0)\big) > 0$ and $\partial^2 f/\partial x^2{}_{(x,y)=(x_0,y_0)} > 0$ then (x_0, y_0) is a *Local Minimum.*

If, on the other hand, $\det(H(x_0, y_0) < 0$ then (x_0, y_0) is a *Saddle Point.*

#include picture

Finally, if det $\big(H(x_0, y_0)\big) = 0$ then the test is inconclusive, and the point (x_0, y_0) could be either a saddle point or a local extremum.

Exercises

Exercise 1 Find and classify all critical points of each of the following functions using the Hessian

$$h_1(x,y) = 28 - 2y^4 - x^2 - 8xy \qquad h_2(x,y) = y^3 - 2x^2 + 12xy + 1$$
$$h_3(x,y) = x^2 + y^2 + \frac{32}{xy} \qquad h_4(x,y) = \exp\big(-(x^2 + y^2 + 4x)\big)$$

5.2 Lagrange Multipliers**

5.3 Multiple Integrals

We perform one *'layer'* of the integration at a time

Exercises

Exercise 1 Calculate each of the following double integrals

$$I_1 = \int_0^1 \int_{x^2}^{\sqrt{x}} (4xy - 1)\, \mathrm{d}y\, \mathrm{d}x \qquad I_2 = \int_{-2}^0 \int_1^2 \big(x^3 - 2x^2 y + y^2\big)\, \mathrm{d}x\, \mathrm{d}y$$

Exercise 2 Calculate each of the following double integrals by changing the order of integration

$$I_1 = \int_0^2 \int_{2x}^4 \big(y^2 \exp(-xy)\big)\, \mathrm{d}y\, \mathrm{d}x \qquad I_2 = \int_0^1 \int_x^1 \big(y^2 \exp(xy)\big)\, \mathrm{d}y\, \mathrm{d}x$$

5.3.1 Coordinate Systems

Cylinderical Polar Coordinates

Spherical Polar Coordinates

Parametric Equations for Curves**

5.3.2 Change of Variables

5.3.3 Jacobians

Exercises

Exercise 1 Use Polar Coordinates to calculate area enclosed by the following curves

$$r_1(\theta) = 4\cos(2\theta) \qquad \text{for } \frac{-\pi}{4} < \theta < \frac{\pi}{4} \text{ and } r_2(\theta) = 1 - \sin(\theta) \qquad \text{for } 0 < \theta < 2\pi \quad (1)$$

5.3.4 Gaussian Integrals****

By Guass,

$$\int_{-\inf}^{\inf} \exp\left(-x^2\right) \mathrm{d}x = \sqrt{\pi}$$

Exercises

Exercise 1 Calculate the Gaussian integral[33]

$$I_1 = \int_{-\inf}^{\inf} \exp\left(-x^2\right) \mathrm{d}x$$

Exercise 2 Calculate each of the following definite integrals, using the result (1)

$$I_1 = \int_{-\inf}^{\inf} \exp\left(-x^2 - ax\right) \mathrm{d}x \qquad I_2 = \int_{-\inf}^{\inf} \exp\left(-\left({}^{x-\mu}/_{\sigma}\right)^2\right) \mathrm{d}x$$

[33] *Hint (due to Poisson):*

$$\left(\int_{-\inf}^{\inf} \exp\left(-x^2\right) \mathrm{d}x\right)^2 = \left(\int_{-\inf}^{\inf} \exp\left(-x^2\right) \mathrm{d}x\right)\left(\int_{-\inf}^{\inf} \exp\left(-y^2\right) \mathrm{d}y\right)$$
$$= \int_{-\inf}^{\inf}\int_{-\inf}^{\inf} \exp\left(-\left(x^2+y^2\right)\right) \mathrm{d}x\,\mathrm{d}y$$

and in polar coordinates

$$x^2 + y^2 = r^2$$

with Jacobian

$$\mathrm{d}x\,\mathrm{d}y = r\mathrm{d}r\,\mathrm{d}\theta$$

6 Linear Algebra

6.1 Basic Properties of Matrices

A *Matrix* is an array consisting of rows and columns of entries. The entries can be numbers, fractions, or even polynomials.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

6.1.1 Addition of Matrices

Addition of matrices is done *element-wise*.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix} \tag{2}$$

This means that we can only add or subtract matrices of the same size.

Addition of matrices is *Commutative*. This means that for two matrices A and B, we get

$$A + B = B + A$$

as we are used to with real numbers.

The *Zero Matrix* 0 is the matrix with 0s everywhere. It has the special property that

$$0 + A = A$$

for any matrix A.

6.1.2 Scalar Multiplication of Matrices

Adding a matrix A to itself repeatedly is the same as multiplying the matrix by a real number

$$A + A + \cdots + A = nA$$

We call this *Scalar Multiplication*.

6.1.3 Multiplication of Matrices

To multiply two matrices we follow the rule *'across and down'*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} \tag{3}$$

and similarly for larger matrices.

Because of the way that this rule works, we cannot multiply any two matrices of arbitrary size together. The number of colmuns of the first matrix must equal the number of rows of the second matrix.

Matrix multiplication is *Associative.* This means that the order of brackets does not matter. That is, for any matrices A, B and C we get

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

Matrix multiplication is not commutative. Thus, in general, for two matrices A and B, we get

$$A \cdot B \neq B \cdot A$$

Thus we must often specify between *Left-Multiplication* and *Right-Multiplication*

The *Identity Matrix* I is the matrix with 1s on the diaginal, and 0s everywhere else. It has the special property that

$$I \cdot A = A \qquad \text{and} \qquad A \cdot I = A$$

for any matrix A

6.2 The Determinant of a Matrix

For a two-by-two matrix

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \tag{4}$$

For larger matrices, we *break the matrix up into smaller parts*, and repeat until we are left with two-by-two matrices. Then we can use our formula (4) to calculate the answer.

To break the matrix up, we

- Starting in the top left-hand corner, *'pull out'* the entry and mutliply it by the determinant of the matrix obtained by covering over the top row and first column

- Move to the next entry to the right, pull it out, and multiply by -1 times the determinant of the the matrix obtained by covering over the top row and second column
- repeat until all entries in the top row have been pulled out in this way

That is

$$\det \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} = a \det \begin{pmatrix} q & r \\ y & z \end{pmatrix} + b \cdot (-1) \det \begin{pmatrix} p & r \\ x & z \end{pmatrix} + c \det \begin{pmatrix} p & q \\ x & y \end{pmatrix}$$

6.2.1 Trace of a Matrix

The *Trace* of a matrix is the sum of its diagonal entries

$$\operatorname{tr} \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} = a + q + z$$

Exercises

[?] §3.1, Exercises 1 to 11.

6.2.2 Rank of a Matrix

6.2.3 Exponentiation of Matrices****

6.3 Types of Matrices

6.3.1 Invertible Matrices

An n-by-n matrix A is an *Invertible Matrix* if

$$\det(A) \neq 0$$

We say that a matrix is *Nonsingular* if it is invertible, and that it's inverse matrix exists.

6.3.2 The Invertible Matrix Theorem‡‡

We have many equivelent conditions for a matrix to be invertible‡

6.3.3 Singular Matrices

On the other hand, if

$$\det(A) = 0$$

we say that the matrix is *Singular*, that is, it is not invertible.

Exercises

Exercise 1 Check if each of the followig matrices are singular or nonsingular

$$(i) \quad A_1 = \qquad\qquad (ii) \quad A_2 =$$

$$(iii) \quad A_3 = \qquad\qquad (iv) \quad A_4 =$$

6.3.4 Diagonal Matrices

An n-by-n matrix D is a *Diagonal Matrix* if it has 0s everywhere except along the diagonal

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

6.3.5 Upper-Triangular and Lower-Triangular Matrices

An n-by-n matrix U is an *Upper-Triangular Matrix* if it has 0s everywhere below the diagonal

On the other hand, an n-by-n matrix L is a *Lower-Triangular Matrix* if it has 0s everywhere above the diagonal

$$U = \begin{pmatrix} a & b & c \\ 0 & q & r \\ 0 & 0 & z \end{pmatrix} \qquad \text{and} \qquad L = \begin{pmatrix} a & 0 & 0 \\ p & q & 0 \\ x & y & z \end{pmatrix}$$

A *Strictly Upper-Triangular Matrix* S_U is one which has 0s everywhere below the diagonal as well as along the diagonal, while a *Strictly Lower-Triangular Matrix* S_L is one which has 0s everywhere above the diagonal as well as along the diagonal

$$S_U = \begin{pmatrix} 0 & b & c \\ 0 & 0 & r \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad S_L = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ x & y & 0 \end{pmatrix}$$

6.3.6 Lower-Diagonal-Upper Decomposition of Matrices‡‡

Exercises

Exercise 1 Decompose the following matrices into the sum of a strictly-lower-triangular, a strictly-upper-triangular and a diagonal matrix

$$(i) \quad A_1 = \qquad (ii) \quad A_2 =$$
$$(iii) \quad A_3 = \qquad (iv) \quad A_4 =$$

6.3.7 Transpose of a Matrix

The *Transpose* of a matrix A is the matrix gotten by *'swapping rows and colmuns'*

$$\begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix}^T = \begin{pmatrix} a & p & x \\ b & q & y \\ c & r & z \end{pmatrix}$$

Exercises

Exercise 1 Calculate the transpose of the follow matrices

$$(i) \quad A_1 = \qquad (ii) \quad A_2 =$$
$$(iii) \quad A_3 = \qquad (iv) \quad A_4 =$$

6.3.8 Symmetric Matrices

An n-by-n matrix A is a *Symmetric Matrix* if

$$A^T = A$$

Exercises

Exercise 1 Check if each of the following matrices are symmetric

$$(i) \quad A_1 = \qquad (ii) \quad A_2 =$$
$$(iii) \quad A_3 = \qquad (iv) \quad A_4 =$$

6.3.9 Orthogonal Matrices

An n-by-n matrix A is an *Orthogonal Matrix* if

$$A^T \cdot A = I$$

That is, A is orthogonal if $A^T = A^{-1}$.

Exercises

Exercise 1 Check if each of the following matrices are orthoganal

$$(i) \quad A_1 = \qquad (ii) \quad A_2 =$$

$$(iii) \quad A_3 = \qquad (iv) \quad A_4 =$$

6.3.10 Nilpotent Matrices

An n-by-n matrix A is *Nilpotent* if it has the property that the result when A is multiplied by itself a finite number of times equals the zero matrix. That is, if

$$A^n = 0$$

for some $n \in \mathbb{N}$.

Clearly all strictly upper- and strictly lower-triangular matrices are nilpotent.

Exercises

Exercise 1 Check if each of the following matrices are nilpotent

$$(i) \quad A_1 = \qquad (ii) \quad A_2 =$$

$$(iii) \quad A_3 = \qquad (iv) \quad A_4 =$$

6.4 Systems of Linear Equations

We may represent systems of simultaneous equations using matrices

$$\begin{aligned} a_1x + b_1y + c_1z &= r_1 \\ a_2x + b_2y + c_2z &= r_2 \\ a_3x + b_3y + c_3z &= r_3 \end{aligned} \quad \Leftrightarrow \quad \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Or, in short

$$A \cdot \vec{x} = \vec{b}$$

where

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \qquad \text{and} \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Here A is the matrix of coefficients, $\vec{x}$ is the vector of unknown variables, while $\vec{b}$ is the vector of known constants.

6.4.1 Row Operations and Gauss-Jordan Elimination

We may perform three kinds of *Row Operations* on matrices

- We may multiply or divide a row by any nonzero number
- We may add or subtract one row to another
- We may swap a pair of rows

Using these operations, we may find *Similar Matrices* to a given matrix A[34].

We say that a matrix A is in *Row Echelon Form* if, reading from the left, the first nonzero entry of a nonzero row is equal to 1 and is always strictly to the right of the first nonzero entry of the row above.

Moreover, we say that a matrix A is in *Reduced Row Echelon Form* if it is in row echelon form, and if, reading from the left, the first nonzero entry of a nonzero row is equal to 1 and if all entries above it are zero[35].

6.4.2 Finding the Inverse of a Matrix

To find the inverse of a nonsingular matrix[36] we augment A with an identity matrix and bring the augmented matrix $\big(A|I\big)$ to reduced row echelon form[37] . Then the resulting augmented matrix will be $\big(I|A^{-1}\big)$.

[34] Recall the high-school method of solving systems of simultaneous equations.
[35] Like many things in maths, some examples will make this much clearer.
[36] Recall that a matrix A is nonsingular if $\det(A) \neq 0$, and in particular, if A^{-1} exists.
[37] Note that for most Linear Algebra calculations there are several ways to do any given computation. Here we stick to row reduction, but other methods are of course acceptable.

6.4.3 Solving Systems of Linear Equations

If a system of linear equations is represented by the matrix equation

$$A \cdot \vec{x} = \vec{b}$$

and if A^{-1} exists, then we can solve for the unknown vector $\vec{x}$ by left-multiplying by A^{-1} to give

$$\vec{x} = A^{-1} \cdot \vec{b}$$

since $A^{-1}A = I$ and $I \cdot \vec{x} = \vec{x}$.

Exercises

[?] §3.2, Exercises 1 to 10.

Exercise 1 Solve the following systems of linear equations by finding the inverse of the matrix of coefficients

$$(i) \quad \begin{aligned} x + 2y &= 3 \\ y &= 7 \\ 3z &= 9 \end{aligned} \qquad\qquad (ii) \quad \begin{aligned} 3x + y + z &= 1 \\ x + 3y + z &= 2 \\ x + y + 3z &= 3 \end{aligned}$$

$$(iii) \quad \begin{aligned} x - 2y + z &= 0 \\ 2y - 8z &= 8 \\ -4x + 5y + 9z &= -9 \end{aligned} \qquad\qquad (iv) \quad \begin{aligned} x + 3y + 4z &= 1 \\ 2x + 6y + 9z &= 2 \\ 3x + y - 2z &= 1 \end{aligned}$$

6.4.4 Solving Systems of Linear Equations with no Feasable Solution

If $A\vec{x} = \vec{b}$ is a system of linear equations with $\det(A) = 0$ then the inverse A^{-1} does not exist, and so we cannot use the above method to solve the system of equations.

We instead introduce an augmented matrix $\big(A|\vec{b}\big)$.

If $\text{rank}(A) < \text{rank}\big(A|\vec{b}\big)$ then the system has *No Feasable Solution.*

Exercises

Exercise 1 Solve the following systems of linear equations by using row reduction

$$(i)\quad \begin{aligned} 3x + 2y + 7z &= 10 \\ x - y + 3z &= 3 \\ 6x + 4y + 14z &= 2 \end{aligned} \qquad\qquad (ii)\quad \begin{aligned} x + y + 3z &= 1 \\ x + 3y + z &= 2 \\ x + y + 3z &= 3 \end{aligned}$$

6.4.5 Solving Systems of Linear Equations with Infinite Solutions

If, on the other hand, $\det(A) = 0$ and $\text{rank}(A) = \left(A|\vec{b}\right)$ then the system will have an infinite number of solutions.

Exercises

Exercise 1 Solve the following systems of linear equations by using row reduction

$$(i)\quad \begin{aligned} x - y + 3z &= 2 \\ 2x - 3y + 9z &= 3 \\ -x + y - 3z &= -2 \end{aligned} \qquad\qquad (ii)\quad \begin{aligned} x + y + 3z &= 6 \\ 2x + 2y + 6z &= 12 \\ 2x + y + 6z &= 11 \end{aligned}$$

6.5 Eigenvalues & Eigenvectors**

6.5.1 Characteristic Equation of a Matrix

6.5.2 Eigenvalues and Eigenvectors

6.5.3 Finding Eigenvalues and Eigenvectors of Invertible Matrices

6.5.4 Systems of Linear Equations with no Feasable Solution

6.5.5 Systems of Linear Equations with Infinite Solutions

6.5.6 Quadratic Forms**

A symmetric matrix A is *Positive-Definite* if all of its eigenvalues are positive.

On the other hand, A is *Negative-Definite* if all of its eigenvalues are negative.

In addition, A is *Positive-Semi-Definite* if all of its eigenvalues are non-negative, while A is *Negative-Semi-Definite* if all of its eigenvalues are non-positive.

Finally, A is *Indefinite* if it has both positive and eigenvalues.

6.5.7 The Jordan Normal Form****

6.6 Vectors**

6.6.1 Vector Addition & Scalar Multiplication‡‡

A *Vector* is a one-dimensional array.

Addition of vectors is done *element-wise*.

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

This means that we can only add or subtract vectors of the same size.

Addition of vectors is *Commutative*. That is, for two vectors $\vec{x}$ and $\vec{y}$, we get

$$\vec{x} + \vec{y} = \vec{y} + \vec{x}$$

as we are used to with real numbers. In particular, this follows from the rules for vector addition and the rules for addition of real numbers.

The *Zero Vector* $\vec{0}$ is the vector of all 0s. It has the special property that

$$\vec{0} + \vec{x} = \vec{x}$$

for any vector $\vec{x}$.

Adding a vector $\vec{x}$ to itself repeatedly is the same as multiplying the vector by a real number

$$\vec{x} + \vec{x} + \cdots + \vec{x} = n\vec{x}$$

We call this *Scalar Multiplication*.

6.6.2 The Norm of a Vector

Given a vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ the *Norm* $|\vec{x}|$ of $\vec{x}$ is given by

$$|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Exercises

Exercise 1 Calculate the norm of the following vectors

(i) $\vec{x} = (3, 4, 1)$	(ii) $\vec{y} = (1, 1, 1)$
(iii) $\vec{x} = (1, -1, 0, 3, 2))$	(iv) $\vec{y} = (0, -1, 1, 1, 2)$
(v) $\vec{x} = (2, 1, 1)$	(vi) $\vec{y} = (2, -3, -1)$
(vii) $\vec{x} = (1, 4, -2)$	

6.6.3 The Dot-Product

Given two vectors $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$ the *Dot-Product* $\vec{x} \cdot \vec{y}$ of $\vec{x}$ and $\vec{y}$ is given by

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

It follows directly that

$$|\vec{x}|^2 = \vec{x} \cdot \vec{x}$$

The dot-product of two vectors is a real number[38].

The dot-product is commutative, that is

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$$

for any vectors $\vec{x}$ and $\vec{y}$.

If θ is the angle that the vectors $\vec{x}$ and $\vec{y}$ make with each other, then

$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos(\theta)$$

Consequently, if $\vec{x}$ and $\vec{y}$ are perpendicular then $\vec{x} \cdot \vec{y} = 0$ since $\cos(\pi/2) = 0$.

Exercises

Exercise 1 Calculate the dot product of the following pairs of vectors

(i) $\vec{x} = (1, -1, 0, 3, 2)$ and $\vec{y} = (0, -1, 1, 1, 2)$ (ii) $\vec{x} = (1, 1,$

(iii) $\vec{x} = (2, -3, -1)$ and $\vec{y} = (1, 4, -2)$ (iv) $\vec{x} = (2, -6, -3)$ and $\vec{y} = (4, 3, -1)$

[38] Note that some books use the term *Scalar-Product* for what we call the dot-product, since real numbers are called *'scalalrs'* to distinguish them from vectors.

6.6.4 The Cross Product

In three dimensions, as well as the dot-product, we may define a second type of vector multiplication.

Given two vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ the *Cross-Product* $\vec{x} \times \vec{y}$ of $\vec{x}$ and $\vec{y}$ is given by

$$\vec{x} \times \vec{y} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

The cross-product of two vectors is itself a vector[39].

The cross-product is *Anti-Commutative*, that is,

$$\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$$

for any vectors $\vec{x}$ and $\vec{y}$.

If θ is the angle that the vectors $\vec{x}$ and $\vec{y}$ make with each other, then

$$\vec{x} \times \vec{y} = |\vec{x}||\vec{y}| \sin(\theta)\vec{n}$$

where $\vec{n}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ and $|\vec{n}| = 1$.

Consequently, if $\vec{x}$ and $\vec{y}$ are parallel then $\vec{x} \times \vec{y} = 0$ since $\sin(0) = 0$.

Exercises

Exercise 1 Calculate the cross product of the following pairs of vectors

(i) $\vec{x} = (2, -6, -3)$ and $\vec{y} = (4, 3, -1)$ (ii) $\vec{x} = (1, 1, 1)$ and $\vec{y} = (2, 1, 1)$

(iii) $\vec{x} = (2, -3, -1)$ and $\vec{y} = (1, 4, -2)$

[39] Note that some books use the term *Vector-Product* for what we call the cross-product.

7 Probability & Statistics***

7.1 Introductory Statistics

Statistics, Mean, Median, Mode, Standard Deviation, Expectation Values,

7.2 Introductory Probability

Probability, Mutually Exclusive Events, Non-Mutually Exclusive Events, Independent Events, Bayes' Theorem, The Binomial Theorem

7.3 Theoretical Probability

Discrete Random Variables, Continuous Random Variables,

The Law of Large Numbers

The Central Limit Theorem

7.4 Discrete Probability Distibrutions

Discrete Probability Distibrutions, Probability Mass Functions,

The Discrete Uniform Distribution, The Poisson Distribution, The Bernoulli Distribution, The Binomial Distribution, The Geometric Distribution

7.5 Continuous Probability Distributions

Continuous Probability Distributions, Cumulative Probability Functions, Probability Density Functions

The Continuous Uniform Distribution, The Normal Distribution

7.6 Hypothesis Testing

Confidence Intervales, Hypothesis Testing

8 Complex Analysis****

8.1 Complex Numbers

Functions of a Single Complex Variable, The Exponential Function revisited, The Logarithm Function revisited, The Trigonometric Functions revisited

8.2 Complex Analysis*****

Holomorphic Functions, Singularities, Branch Points, Meromorphic Functions, The Laurent Series, Radius of Convergence

The Cauchy-Riemann Equations, Analytic Functions

The Louville Theoerm, The Picard Theorem

The Cauchy Integral Theorem

The Cauchy Integral Formula

The Cauchy Residue Theorem

Contour Integrals

9 Fourier Series****

9.1 Fourier Series

Calculating some values of the Riemann zeta-Function

9.2 The Complex Fourier Series

9.3 Fourier Transform

9.4 Distributions*****

The Dirac delta-Function

The Gauss gamma-Function

10 Vector Calculus****

10.0.1 Vector Valued Functions

10.0.2 Parameterised Curves

10.1 Differentiation of Vector Valued Functions

10.1.1 Directional Derivatives

10.1.2 Gradient, Divergence & Curl

Path Independence, Conservative Vector Fields, Potential Functions

10.2 Integration of Vector Valued Functions

10.2.1 Line Integrals

10.2.2 Surface Integrals

10.2.3 The Green Theorem

10.2.4 The Gauss Divergence Theorem

10.2.5 The Stokes Thoerem

10.3 The General Stokes Theorem*****

The General Stokes Theorem, The Fundamental Theorem of Calculus from the General Stokes Theorem, The Gauss Divergence Theorem from the General Stokes Theorem, The Stokes Theorem from the General Stokes Theorem

11 Partial Differential Equations****

11.1 Boundary Conditions

Direchlet Boundary Conditions, Von Neumann Boundary Conditions

11.2 The Method of Characteristics

11.3 The Heat Equation

11.4 The Wave Equation

11.5 The Laplace Equation

12 Algebra*****

12.1 Set Theory

Introductory Set Theory

Maps Between Sets, Image of a map, Kernal of a map

12.2 Group Theory

Introductory Group Theory, Homomorphism, Isomorphism, Abelian Groups, Lagrange's Theorem, Quotient Groups, Integers Modulo n, The Fundamental Theory of Abelian Groups

12.3 Ring Theory

Rings, Ideals, Commutative Rings, Integral Domains, Unique Factorisation Domains, Principal Ideal Domains, Division Rings,

Rings of Polynomial Functions, Minimal Polynomials, The Gaussian Numbers and Guassian Integers

12.4 Field Theory

Fields, Characteristic of a Field, Prime Subfields, Algebraic Closure, Finite Field Extensions

12.5 Number Theory

Introductory Number Theory, The Infinity of Primes, The Riemann zeta-Function revisited, The Euler Totient Function

12.6 Algebraic Number Theory

Introductory Algebraic Number Theory, Algebraic Numbers, Algebraic Integers, Algebraic Number Fields, Rings Algebraic Integers, Integral Closure, Quadratic Number Fields, Rings of Quadratic Integers, Gaussian Primes, Sums of Squares, Quadratic Reciprocity, Cyclotomic Fields