24/03/2015 MA1124 # Analysis - Homework 9

1. - We wish to prove the Bolzano-Wéierstrass theorem given that every Cauchy sequence $(x_n)_{n=0}^{\infty}$ in $\mathbb{R}^n$ converges

~ So, we to prove that if $(x_n)_{n=0}^{\infty}$ is a bounded sequence in $\mathbb{R}^n$ then there exists a convergent subsequence $(x_{n_i})_{i=0}^{\infty}$

~ We recall that $(x_n)_{n=0}^{\infty}$ is Cauchy if given
$$\varepsilon > 0$$
there exists some $N \in \mathbb{N}$ such that for all
$$n > N \quad \text{and} \quad m > N$$
we get
$$|x_n - x_m| < \varepsilon$$

~ Moreover, $(x_n)_{n=0}^{\infty}$ converges to $x \in \mathbb{R}^n$ such that
$$\lim_{n\to\infty} x_n = x$$
if given
$$\delta > 0$$
there exists some $M \in \mathbb{N}$ such that
$$|x_n - x| < \delta$$
for all
$$n > M$$

~ Now, if $(x_n)_{n=0}^{\infty}$ is bounded then an upper bound for $(x_n)_{n=0}^{\infty}$ exists and a lower bound for $(x_n)_{n=0}^{\infty}$ exists

~ In particular, $\inf_n(x_n)$ exists and $\sup_n(x_n)$ exists, with

$$\inf_n(x_n) = a \quad \text{and} \quad \sup_n(x_n) = b$$
say, by the greatest lower bound principle and

the least upper bound principle respectively

~ In addition, we recall that the Nested Interval theorem states that if $I_1, I_2, \ldots$ is a countably infinite sequence of nested intervals such that

$$I_1 \supset I_2 \supset \ldots$$

with each I_j a closed bounded interval then

$$\cap_{n=1}^{\infty} I_i \neq \phi$$

~ Well, we have $x_n \in [a,b]$ for all $n \in \mathbb{N}$

~ Let

$$\Theta_1 = [a,b]$$

and let $y_1 \in \Theta_1$

~ Then let

$$U_{2_1} = [a, \tfrac{1}{2}(a+b)] \quad \text{and} \quad U_{2_2} = [\tfrac{1}{2}(a+b), b]$$

~ But there exists an accumulation point y_2, say, such that either $y_2 \in U_{2_1}$ or $y_2 \in U_{2_2}$

~ Then put

$$\Theta_2 = \begin{cases} U_{2_1} & \text{if } y_2 \in U_{2_1} \\ U_{2_2} & \text{if } y_2 \in U_{2_2} \end{cases}$$

~ Similarly, let

$$U_{3_1} = [a, \tfrac{1}{4}(a+b)], \quad U_{3_2} = [\tfrac{1}{4}(a+b), \tfrac{1}{2}(a+b)]$$

and

$$U_{3_3} = [\tfrac{1}{2}(a+b), \tfrac{3}{4}(a+b)] \quad \text{and} \quad U_{3_4} = [\tfrac{3}{4}(a+b), b]$$

~ Then there exists an accumulation point y_3, 24/03/2015 MA 1124 Analysis - Homework 9

1. say, such that either $y_3 \in U_{3_1}$ or $y_3 \in U_{3_2}$ or $y_3 \in U_{3_3}$ or $y_3 \in U_{3_4}$

~ Then put

$$O_3 = \begin{cases} U_{3_1} & \text{if } y_3 \in U_{3_1} \\ U_{3_2} & \text{if } y_3 \in U_{3_2} \\ U_{3_3} & \text{if } y_3 \in U_{3_3} \\ U_{3_4} & \text{if } y_3 \in U_{3_4} \end{cases}$$

~ Likewise, let

$$U_{n_1} = \left[a, \frac{1}{2^{n-1}}(a+b)\right], \quad U_{n_2} = \left[\frac{1}{2^{n-1}}(a+b), \frac{2}{2^{n-1}}(a+b)\right], \ldots$$

and

$$U_{n_{2^{n-1}}} = \left[\frac{2^{n-1}-1}{2^{n-1}}(a+b), b\right]$$

such that

$$U_{n_k} = \left[\frac{k-1}{2^{n-1}}(a+b), \frac{k}{2^{n-1}}(a+b)\right]$$

for each

$$k = 2, 3, \ldots, 2^{n-1}-1$$

~ Then, again, there exists an accumulation point y_n, say, such that either $y_n \in U_{n_1}$ or $y_n \in U_{n_2}$ or ... or $y_n \in U_{n_{2^{n-1}}}$

~ Then put

$$O_n = \begin{cases} U_{n_1} & \text{if } y_n \in U_{n_1} \\ U_{n_2} & \text{if } y_n \in U_{n_2} \\ \vdots & \vdots \\ U_{n_{2^{n-1}}} & \text{if } y_n \in U_{2_{2^{n-1}}} \end{cases}$$

~ But $O_n \subset B(y_n, r_n)$ with

$$r_n = 2 \cdot \frac{1}{2^{n-1}}(a+b) = \frac{1}{2^{n-2}}(a+b)$$

for each n

~ However

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{1}{2^{n-2}}(a+b) = 0$$

~ Hence given $\varepsilon > 0$

there exists $N \in \mathbb{N}$ such that for all $n > N$ and $m > N$

we get

$$|y_n - y_m| < \underbrace{|r_n - r_m|}_{<\varepsilon}$$

~ Thus $(y_n)_{n=1}^{\infty}$ is Cauchy

~ Accordingly $(y_n)_{n=1}^{\infty}$ is convergent ∎

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2. - We wish to prove the Heine-Borel theorem given that every Cauchy sequence $(x_n)_{n=0}^{\infty}$ in $\mathbb{R}^n$ converges

~ So, we wish to prove that if $A \subset \mathbb{R}^n$ is a closed and bounded set then A is compact

~ Well, if A is bounded then an upper bound for A exists and a lower bound for A exists

~ In particular, $\inf(A)$ exists and $\sup(A)$ exists by the greatest lower bound principle and by the least upper bound principle respectively, with

$$\inf(A) = a \quad \text{and} \quad \sup(A) = b$$

say

~ Moreover, if A is a closed set then $\inf(A) \in A$ and $\sup(A) \in A$

~ Now, A is compact if every open cover $\mathcal{U}$ of A has a finite subcover

~ Let $\mathcal{U}$ not have a finite subcover

~ Put

$$\mathcal{O}_1 = [a, b]$$

~ Let

$$\mathcal{U}_{2_1} = [a, \tfrac{1}{2}(a+b)] \quad \text{and} \quad \mathcal{U}_{2_2} = [\tfrac{1}{2}(a+b), b]$$

~ Then put

- $$\mathcal{O}_2 = \begin{cases} U_{2_1} & \text{if there does not exist a finite subcover for } U_{2_1} \\ U_{2_2} & \text{if there does not exist a finite subcover for } U_{2_2} \end{cases}$$

~Similarly, let
$$U_{n_1} = \left[a, \frac{1}{2^{n-1}}(a+b)\right], U_{n_2} = \left[\frac{1}{2^{n-1}}(a+b), \frac{2}{2^{n-1}}(a+b)\right], \ldots$$
and
$$U_{n_{2^{n-1}}} = \left[\frac{2^{n-1}-1}{2^{n-1}}(a+b), b\right]$$

~Then put
$$\mathcal{O}_n = \begin{cases} U_{2_1} & \text{if there does not exist a finite subcover for } U_{2_1} \\ U_{2_2} & \text{if there does not exists a finite subcover for } U_{2_2} \\ \vdots & \vdots \\ U_{2_{2^{n-1}}} & \text{if there does not exist a finite subcover for } U_{2_{2^{n-1}}} \end{cases}$$

~Then
$$\mathcal{O}_1 \supset \mathcal{O}_2 \supset \ldots \supset \mathcal{O}_n$$
with
$$\mathcal{O}_k \neq \emptyset$$
for each
$$k = 1, 2, \ldots, n$$
by the Nested Interval theorem

~Thus letting $y_k \in \mathcal{O}_k$ for each
$$k = 1, 2, \ldots, n$$
we see that $\mathcal{O}_k \subset B(y_k, r_k)$ with

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2.

$$r_k = 2 \cdot \frac{1}{2^{k-1}}(a+b)$$
$$= \frac{1}{2^{k-2}}(a+b)$$

and so given

$$\varepsilon > 0$$

there exists $N \in \mathbb{N}$ such that for all

$n > N$ and $m > N$

we get

$$|y_n - y_m| \leq |r_n - r_m|$$
$$< \varepsilon$$

since

$$\lim_{k \to \infty} r_k = \lim_{k \to \infty} \frac{1}{2^{k-2}}(a+b)$$
$$= 0$$

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~ Hence $(y_n)_{n=1}^{\infty}$ is Cauchy

~ Accordingly $(y_n)_{n=1}^{\infty}$ converges, with

$\lim_{n \to \infty} y_n = l$

say

~ But then $B(l, \delta)$ is a finite subcover for O_n with

$$\delta > r_n$$

~ However this is a contradiction

~ Therefore A is compact

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