The Cosmological Chameleon

A Scalar-Tensor Theory of Gravity & Dark Energy

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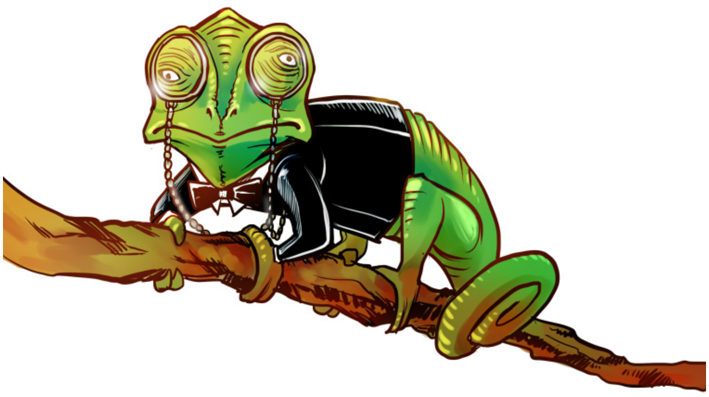

Contents

Abstract

Recent measurements of Type Ia Supernovae indicate that the universe is currently undergoing a period of accelerated cosmic expansion. We are thus directed to the conclusion that our universe is filled with an elusive *Dark Energy*. While the standard model of the dark energy is that of a *Cosmological Constant*, the hunt for an explanation of such effects has led to the proposal of a *Fifth Force*, often dubbed *Quintessence*. The usual quintessence theory consists of a slim scalar field which couples to all matter fields. However, a novel proposal has been made in which this field has a mass which is a function of the ambient background density, and is called a *Chameleon* scalar field.

In this exposition the scalar-tensor theory of gravity with a chameleon field is reviewed. The inflationary and cosmological evolution of the chameleon field is investigated. It is found that equipartition after reheating with a standard inflaton scalar field leads to a chameleon field which overshoots the minimum of its effective potential. As each Standard Model particle species becomes non-relativistic during the evolution of the early universe the chameleon field moves towards the minimum of its effective potential energy function, which it must reach before the beginning of Big Bang Nucleosynthesis. This is necessary in order to conform to experimental bounds on the possible change in the mass of the elementary particles since the time of Big Bang Nucleosynthesis. This minimum value is an attractor solution which the chameleon field then follows until today. In the late universe, for a runaway potential, this gives rise to an additional constant term in the Einstein field equations which plays the rôle of the cosmological constant. That is, it gives accelerated cosmic expansion.

The effect of the chameleon field on the cosmic microwave background anisotropies due to a *Chameleonic Sunyaev-Zel'Dovich Effect* is then discussed. The cosmic microwave background is introduced and described in brief. So too are two models for the magnetic field of a galaxy cluster introduced, the *Cell Model* and the Power Spectrum Model respectively. This is followed by a short account of the *Thermal Sunyaev-Zel'Dovich Effect*. Finally the chameleonic modifications to the cosmic microwave background anisotropies as a possibility for testing the theory will be explored. It is found that using the *Coma* cluster as an example to study a bound of $M_F > 1.1 \times 10^9$GeV on the scale of the chameleon theory is obtained.

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Chapter 1

Introduction

1.1 The Road To A Fifth Force

Our universe is governed by four fundamental forces, which in order of elucidation are the *Gravitational Force*, the *Electromagnetic Force*, the *Weak Interactions*, and the *Strong Interactions*. The gravitational and electromagnetic forces are long range, and permeate our lives. Thus we are most familiar with them. The gravitational force was first put on firm footing by Newton in 1678[1], and brought into its modern relativistic formulation by Einstein in the 1910s[2, 3].

The first *Physical Cosmology* was that of a *Static Einstein Universe*[4]. This consisted of a temporally infinite but spatially finite flat universe with a positive cosmological constant Λ_E given by

$$\Lambda_E = \frac{4\pi G\rho}{c^2} \tag{1.1}$$

where ρ is the matter density, G is the Newtonian gravitational constant and c is the limiting speed of *Special Relativity*[5]. This cosmological constant causes a repulsive force so as to overcome the gravitational attraction of distant massive bodies. Hence the universe is kept in an unstable stationary state.

This was followed by the *de Sitter Universe* later the same year. Here the universe is spatially flat and has no matter content, but does have a positive cosmological constant Λ. This causes constant exponential expansion, that is, *Inflation*, with an expansion rate

$$H \propto \sqrt{\Lambda} \tag{1.2}$$

A number of years later, in 1922, came the positive curvature spherical *Friedmann*

Universe[6]. This idea was extended in 1924[7] to allow for the case of a universe with negative curvature.

In 1927 the first *Big Bang Theory*[8, 9] was proposed by Lemaître, based on the Friedmann universe. This *Friedmann-Lemaître Model* has a positive cosmological constant to give a universe which expands from an initial dense state.

Through the measurement of Cepheid variable stars in distant galaxies, in 1929 Hubble[10] determined that in fact we live in an expanding universe. Explicitly, the velocity v of a distant galaxy is given by *Hubble's Law*

$$v = H_0 d \tag{1.3}$$

where $H_0 \simeq 10^{-33}$eV is the *Hubble Constant* today, and d is the distance to the galaxy.

The Friedmann-Lemaître model was further extended by Robertson and Walker in 1935[11, 12, 13, 14]. These *Friedmann-Lemaître-Robertson-Walker Universes* are isotropic, homogeneous and uniformly expanding.

Recent measurements of Type Ia Supernovae[15, 16] indicate the the universe is currently undergoing a period of accelerated cosmic expansion. The search for an explanation for such effects has led to the proposal of a *Fifth Force*[17, 18, 19, 20], usually dubbed *Quintessence*. If this fifth force is to explain the accelerating cosmic expansion then it too must permeate all of space and time.

The usual quintessence theory consists of a slim scalar field which couples to all matter fields. Experimental searches for a fifth force however have thus far found no deviation to the Equivalence Principle. This places many constraints on the existence of such a slim field which couples to the standard model matter fields gravitationally.

A novel proposal has been made by Khoury and Weltman[21, 22] in which this scalar field has a mass which is a function of the ambient background density. Thus, on Earth, where the density of matter is large, the field would have a large mass term, m_ϕ, and so not contradict current local tests of gravity. However, on the vast cosmological scale the field would be slim, with a mass term satisfying $m_\phi \simeq H_0$ where H_0 is the Hubble constant today. Hence it would be a rolling scalar field. Therefore a linearisation of its potential function would give a constant term of the same scale as the cosmological constant today. That is, the field changes in accordance with its background, and so is called a *Chameleon* scalar field*. This model shall be the focus

*It should be noted that chameleon lizards do not in fact change their colour in order to match their background, contradictory to common perception. Rather, they do this in response to their physical and emotional state[23] of this essay.

1.2 Quintessence

We first briefly discuss the usual quintessence theory. Here[24] an additional dynamical scalar field gives the small vacuum energy density. This field is taken to have a long time attractor solution[25, 26]. In this case, a rolling scalar field is not sensitive to the initial conditions. Such a potential may be described by that of the *Ratra-Peebles Model*[27, 28]. In order to have a rolling field today we must have

$$m_\phi \simeq H_0$$

today. Then the equation of state for the quintessence field would be different from that of a cosmological constant. Consequently, the quintessence must be a massless scalar field. This produces a contradiction, since gravitational tests of the Equivalence Principle give highly constrained bounds on the interaction of a massless quintessence field with the usual Standard Model matter fields.

Chapter 2

The Chameleon Model

2.1 The Chameleon Model

We wish to consider the specifics of the chameleon scalar field model. In doing so we follow [29] closely throughout. We will discuss the action of the theory, and the equations of motion that arise from it. Next we will show how the theory may avoid the bounds set on the quintessence theory by gravitational experiments. Following this, we will show that for a general class of potential energy functions the chameleon field can indeed give rise to late universe accelerated cosmic expansion. Finally we investigate the evolution of the universe with a chameleon field. This gives rise to some limits on the possible initial condition of the chameleon field, namely that it must reach the minimum of its effective potential function before the start of Big Bang Nucleosynthesis. As well as this, the inflationary epoch of the very early universe with a chameleon field is explored. In doing so we follow the model of a scalar inflaton field.

To begin, we have the following scalar-tensor action for the chameleon field

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{M_{\mathrm{Pl}}^2}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) - \int \mathrm{d}^4x\,\mathcal{L}_m(\psi_m^{(i)}, g_{\mu\nu}^{(i)}) \tag{2.1}$$

where

$$\begin{aligned} M_{\mathrm{Pl}} &= \frac{1}{\sqrt{8\pi G}} \\ &\simeq 10^{18}\mathrm{GeV} \end{aligned} \tag{2.2}$$

is the *Reduced Planck Mass* in natural units such that $h = c = 1$ with $\mathcal{L}_m$ the Lagrangian for the matter fields $\psi_m^{(i)}$.

We assume a conformal coupling of the chameleon field to the matter fields $\psi_m^{(i)}$ of species i. That is, the chameleon field gives rise to a fifth force. Then

$$g_{\mu\nu}^{(i)} = \exp\left(\frac{2\beta_i\phi}{M_{\rm Pl}}\right) \tag{2.3}$$

where $g_{\mu\nu}$ is the Einstein frame metric tensor, with β_i being some constant. This gives a Klein-Gordon field equation

$$\nabla^2\phi = V_{,\phi}(\phi) - \sum_i \frac{\beta_i}{M_{\rm Pl}} \exp\left(\frac{4\beta_i\phi}{M_{\rm Pl}}\right) g_{\mu\nu}(i) T_{\mu\nu}^{(i)} \tag{2.4}$$

where

$$T_{\mu\nu}^{(i)} = \frac{2}{\sqrt{-g}^{(i)}} \frac{\delta\mathcal{L}_m}{\delta g_{(i)}^{\mu\nu}}$$

For relativistic degrees of freedom we may let

$$T^{\mu}{}_{\mu} = 0$$

However, when a particle species becomes non-relativistic during the evolution of the early universe we get

$$T^{\mu}{}_{\mu} \neq 0$$

for about one e-fold of expansion.

We may approximate non-relativistic matter of energy density $\tilde{\rho}_i$ as that of a perfect fluid such that

$$g_{(i)}^{\mu\nu} T_{\mu\nu}^{(i)} = -\tilde{\rho}_i$$

where

$$\rho_i = \tilde{\rho}_i \exp\left(\frac{3\beta_i\phi}{M_{\rm Pl}}\right)$$

is conserved in the Einstein frame.

Hence our Klein-Gordon field equation becomes

$$\nabla^2\phi = V_{,\phi}(\phi) + \sum_i \frac{\beta_i}{M_{\rm Pl}} \rho_i \exp\left(\frac{\beta_i\phi}{M_{\rm Pl}}\right) \tag{2.5}$$

and we see that we have an effective potential energy function

$$V_{\rm Eff}(\phi) = V(\phi) + \sum_i \rho_i \exp\left(\frac{\beta_i\phi}{M_{\rm Pl}}\right) \tag{2.6}$$

which is a combination of the scalar field potential and a term proportional to the matter energy-density such that

$$V_{\rm Eff} = V_{\rm Eff}(\rho_i)$$

Now, if either the potential $V(\phi)$ is monotonically decreasing with ϕ and $\beta_i > 0$, or, if $V(\phi)$ is monotonically increasing with ϕ and $\beta_i < 0$, then we see that the effective potential $V_{\rm Eff}(\phi)$ has a minimum for some value $\phi_{\rm min}$ of the chameleon field such that

$$\begin{aligned} V_{{\rm Eff},\phi}\,(\phi_{\rm min}) &= V_{,\phi}(\phi_{\rm min}) + \sum_i \frac{\beta_i}{M_{\rm Pl}} \rho_i \exp\left(\frac{\beta_i \phi_{\rm min}}{M_{\rm Pl}}\right) \\ &= 0 \end{aligned} \tag{2.7}$$

For small fluctuations about the minimum of the effective potential we get a mass term

$$\begin{aligned} m_\phi^2 &= V_{{\rm Eff},\phi\phi}\,(\phi_{\rm min}) \\ &= V_{,\phi\phi}(\phi_{\rm min}) + \sum_i \frac{\beta_i^2}{M_{\rm Pl}^2} \rho_i \exp\left(\frac{\beta_i \phi_{\rm min}}{M_{\rm Pl}}\right) \end{aligned} \tag{2.8}$$

We let $V(\phi)$ be a runaway Ratra-Peebles potential such that

$$V(\phi) = M^4 f\left(\frac{\phi}{M}\right) \tag{2.9}$$

say, where M is the scale of the interaction, and f is some function which gives a quintessence potential with a tracker solution. Thus we impose a tracker solution such that Γ defined as

$$\Gamma = \frac{V_{,\phi\phi}(\phi) V(\phi)}{V_{,\phi}^2(\phi)} \tag{2.10}$$

satisfies $\Gamma > 1$. In addition, we impose that the potential diverge at some finite value of ϕ,

$$\phi = \phi_*$$

say. That is, we impose that

$$\frac{f''\left(\frac{\phi}{M_{\rm Pl}}\right) f\left(\frac{\phi}{M_{\rm Pl}}\right)}{\left(f'\left(\frac{\phi}{M_{\rm Pl}}\right)\right)^2} > 1 \tag{2.11}$$

and

$$\lim_{x \to x_*} f(x) \to \infty \tag{2.12}$$

In what follows we let

$$\phi_* = 0$$

Therefore both the minimum ϕ_{min} and the mass m_ϕ are functions of the background matter density ρ_m. Moreover, m_ϕ is an increasing function of ϕ. Thus, for large matter density ρ_m we get a large field value ϕ, which in turn gives a large chameleon mass m_ϕ. That is, the physical properties of the chameleon field vary with the environment.

2.2 The Thin-Shell Effect

We next consider the *Thin-Shell Effect* of the interaction of the chameleon field with a large massive object. This mechanism allows the chameleon field to go undetected thus far during gravitational experiments which search for a fifth force performed on Earth. Here we follow the approach of [30].

We note the non-linearity of the Klein-Gordon equation (2.4). Moreover, we consider a potential energy function of the form

$$V(\phi) = M^4 \exp\left(\frac{M^n}{\phi^n}\right) \tag{2.13}$$

such that the coupling of the chameleon field to matter given by the exponential function. We let β be of order one, let the chameleon field be finite valued at the origin, and let

$$\phi = \phi_\infty$$

say, at far from the object. Then inside the large massive object there is a minimum value of the effective potential energy function for some value of the chameleon field ϕ_C, say, such that equation (2.7) holds. However, at some finite thickness from the surface of the object ΔR, say, the value of the chameleon field satisfies

$$\phi > \phi_C$$

Furthermore, outside the object we get

$$\phi \propto \frac{1}{r} \exp(-m_0 r)$$

where m_0 is the mass of the chameleon field at far from the object. This implies that

$$\frac{\Delta R}{R} = \frac{\phi_\infty - \phi_C}{6\beta M_{\rm Pl}\Phi_N} \tag{2.14}$$

where

$$\Phi_N = \frac{M}{8\pi M_{\rm Pl}^2 R}$$

is the Newtonian gravitational potential energy. Hence outside the object we find that

$$\phi(r) \simeq -\left(\frac{\beta}{4\pi M_{\rm Pl}}\right)\left(\frac{3\Delta R}{R}\right)\frac{M}{r}\exp\left[-m_0(r-R)\right] \tag{2.15}$$

In order for the shell to be thin we must have

$$\frac{\Delta R}{R} \ll 1$$

Therefore, by equation (2.14) we must have the Newtonian potential Φ_N very large. So, by equation (2.15) we get a correction to the Newtonian gravitational force given by

$$F = (1+\theta)F_N$$

where F_N is the usual Newtonian gravitational force, with

$$\theta = 2\beta^2\left(\frac{3\Delta R}{R}\right)$$

Thus for a small value of θ we see that we only get a small alteration to the Newtonian gravitational force. However this occurs exactly when the shell is thin. Then this in turn occurs when the Newtonian gravitational potential Φ_N is very large, as shown above, which is the case for a large massive object. Therefore for a large massive object there will only be a small deviation from the Newtonian gravitational force F_N.

2.3 Experimental Constraints

In this short section, we shall show that the bounds given by gravitational experiments allow us to set a useful upper-bound on the scale of the chameleon theory.

For a scalar field fifth-force interaction we get a *Yukawa Potential*

$$U(r) = \alpha G M_1 M_2 \frac{\exp\left(\frac{-r}{\lambda}\right)}{r} \tag{2.16}$$

By experimental constraint we have no fifth force which has range λ satisfying $\lambda \gtrsim 100\,\mu$m, for α of order one.

To conform to the bounds set by vacuum chamber experiments we need the range of the chameleon interaction in the atmosphere of the Earth to satisfy $m_{\rm atmo}^{-1} \lesssim 1\,$mm. If this were not the case existing experiments would have been able to find evidence of the interaction between the chameleon field and the usual Standard Model matter fields. However thus far no such fifth-force interactions have been found.

For a potential energy function of the form given by equation (2.13) this leads to a constraint on the scale of the chameleon interaction of

$$M \lesssim 10^{-3}\mathrm{eV} \tag{2.17}$$

This bound on the scale of the chameleon theory will prove useful in much of the discussion that follows.

2.4 Fiducial Potential

We now consider the implications of the particular choice of a general runaway potential energy function for the chameleon field theory. We will find that indeed the chameleon field, similarly to the quintessence scalar field, can give rise to a vacuum energy density that is tiny compared to the Planck mass scale. In addition we will obtain some useful relations, including the fact that the value of the chameleon field when at the minimum of the effective potential is much less than the Planck mass scale for all times since the Hot Big Bang until today.

For a quintessence theory a potential energy function of the form given by equation (2.13) above is usually chosen. Then expanding this potential we find that both locally and on cosmological scales today

$$V(\phi) \simeq M^4 + M^4\left(\frac{M^n}{\phi^n}\right) + \ldots \tag{2.18}$$

Therefore with $M \lesssim 10^{-3}$eV as given by the bound (2.17) we get a constant term

$$M^4 \simeq 10^{-12}\mathrm{eV} \tag{2.19}$$

of the same order as the vacuum energy today. This constant term however is negligible when considering gravitational tests.

We consider only a single matter particle species. Differentiating the potential (2.13) gives

$$V'(\phi) = -nM^n\phi^{-(n+1)}V(\phi)$$

Then substituting this into equation (2.7) we find that at the minimum of the effective potential

$$\left(\frac{M}{\phi_{\min}}\right)^{n+1} = \frac{1}{V(\phi_{\min})}\frac{\beta}{n}\frac{M}{M_{\rm Pl}}\rho_m \exp\left(\frac{\beta\phi_{\min}}{M_{\rm Pl}}\right) \tag{2.20}$$

In addition, differentiating the potential (2.13) twice and substituting into equation (2.8) we see that the mass term is

$$m_\phi^2 = \frac{\beta\rho_m}{MM_{\rm Pl}}\exp\left(\frac{\beta\phi_{\min}}{M_{\rm Pl}}\right)\left[n\left(\frac{M}{\phi_{\min}}\right)^{n+1} + (n+1)\left(\frac{M}{\phi_{\min}}\right) + \beta\frac{M}{M_{\rm Pl}}\right] \tag{2.21}$$

We have from equation (2.18) that

$$V(\phi_{\min}) \simeq M^4 \tag{2.22}$$

Also

$$\rho_m \exp\left(\frac{\beta\phi_{\min}}{M_{\rm Pl}}\right) \simeq M^4$$

Then substituting these into equation (2.20) above, we see that today the value of the chameleon field at the minimum of the effective potential satisfies

$$\phi_{\min}^{(0)} \simeq \left(\frac{\beta}{n}\right)^{\frac{1}{n+1}}\left(\frac{M}{M_{\rm Pl}}\right)^{\frac{n}{n+1}} M_{\rm Pl}$$

or

$$\phi_{\min}^{(0)} \simeq \left(\frac{M}{M_{\rm Pl}}\right)^{\frac{n}{n+1}} M_{\rm Pl} \tag{2.23}$$

for both β and n of order one. Furthermore, since $M \ll M_{\rm Pl}$ equation (2.23) yields

$$\phi_{\min}^{(0)} \ll M_{\rm Pl} \tag{2.24}$$

We note that $\phi_{\min}$ is an increasing function of time, since $\phi_{\min} \propto \rho_m^{-1}$, while $\rho_m \propto t^{-1}$. Thus we find that equation (2.24) holds for all relevant times since the Hot Big Bang until today.

Consequently we can approximate equation (2.21) as

$$m_\phi^2 = \frac{\beta\rho_m}{\phi_{\min}M_{\rm Pl}}\exp\left(\frac{\beta\phi_{\min}}{M_{\rm Pl}}\right)\left[1 + n + n\left(\frac{M}{\phi_{\min}}\right)^{n}\right] \tag{2.25}$$

on ignoring the $\beta \frac{M}{M_{\text{Pl}}}$ term.

We wish to find the conditions that give

$$\phi_{\text{min}} \simeq M$$

Using the approximation (2.22) we get the critical matter density ρ_{Crit} which gives this condition by using equation (2.20) to be

$$\rho_{\text{Crit}} \simeq \frac{M_{\text{Pl}}}{M} \frac{n}{\beta} M^4 \tag{2.26}$$

when $\phi_{\text{min}} \simeq M$. Then letting both β and n be of order one, with $M \simeq 10^{-3}$eV by the bound (2.17), we see that

$$\rho_{\text{Crit}} \simeq 10^{-12}\text{eV}$$

This corresponds to a temperature T_{Crit} of

$$T_{\text{Crit}} \simeq 10\,\text{MeV} \tag{2.27}$$

and a fractional matter energy density Ω_m of

$$\Omega_m \simeq 10^{-6} \tag{2.28}$$

where Ω_m is defined as

$$\Omega_m = \rho_m \exp\left(\frac{\beta \phi_{\text{min}}}{M_{\text{Pl}}}\right) \frac{1}{3H^2 M_{\text{Pl}}^2} \tag{2.29}$$

This critical temperature will aid our later discussion of the equation of state of the chameleon field.

2.5 Cosmological Evolution

We now develop the standard cosmological model which will guide our discussion for the remainder of this chapter.

To this end, we invoke the Cosmological Principle, such that we assume a flat, isotropic, homogeneous universe. Then our metric $g^{\mu\nu}$ becomes the usual Friedmann-Lemaître-Robertson-Walker metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2\right] \tag{2.30}$$

This reduces our Klein-Gordon field equation (2.4) to

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi}(\phi) - \frac{\beta}{M_{\rm Pl}}\rho_m \exp\left(\frac{\beta\phi}{M_{\rm Pl}}\right) \tag{2.31}$$

We recall that the stress-energy-momentum tensor for a scalar field ϕ is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right]$$

Then taking the 00-component gives the energy density of the chameleon field

$$\begin{aligned}\rho_\phi &= \partial^0\phi\partial_0\phi - \delta_0^0\left[\frac{1}{2}\left(\partial_0\phi\partial_0\phi - \partial_i\phi\partial_i\phi\right) - V(\phi)\right]\\ &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\end{aligned} \tag{2.32}$$

since $\phi = \phi(t)$ only. Adding the energy density of the matter fields ρ_m and the radiation energy density ρ_r the usual Friedmann equation

$$\dot{a}^2(t) = \frac{8\pi G}{3}\rho a^2(t) \tag{2.33}$$

becomes

$$3H^2M_{\rm Pl}^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{\beta}{M_{\rm Pl}}\exp\left(\frac{\beta\phi}{M_{\rm Pl}}\right)\rho_m + \rho_r \tag{2.34}$$

Finally, we recall that

$$\frac{\dot{\rho}_m}{\rho_m} = -3H \tag{2.35}$$

and

$$\frac{\dot{\rho}_r}{\rho_r} = -4H \tag{2.36}$$

2.6 Attractor Solution

In the next six sections we develop the consequences of the attractor solution of the quintessence theory in the framework of the chameleon field theory. Here the value of the scalar field ϕ follows the minimum of the effective potential energy function $V_{\rm Eff}$ as it changes during the evolution of the universe, whence

$$\phi_{\rm min} = \phi_{\rm min}(t) \tag{2.37}$$

We first consider the case in which the initial value of the chameleon field ϕ_i is

$$\phi_i = \phi_{\min} \tag{2.38}$$

Then as the matter density decreases the minimum of the effective potential energy function shifts to larger and larger values of the chameleon field. This shift occurs on a timescale of about the Hubble time, H^{-1}.

If the chameleon field undergoes small oscillations with a period of m_ϕ^{-1} about the minimum of the effective potential, then we find the ratio of these two timescales by equation (2.25) to be

$$\frac{m_\phi^2}{H^2} \simeq 3\beta\Omega_m \frac{M_{\rm Pl}}{\phi_{\min}} \left[1 + n + n\left(\frac{M}{\phi_{\min}}\right)^n\right] \tag{2.39}$$

If $m_\phi \gg H$ such that $m_\phi^{-1} \ll H^{-1}$ the the chameleon field will evolve adiabatically as the minimum $\phi_{\min}$ of the effective potential energy function changes.

If on the other hand $m_\phi \ll H$ such that $m_\phi^{-1} \gg H^{-1}$ we find that the chameleon field will lag behind the minimum of the effective potential.

However, if we consider the two cases of $\phi \lesssim M$ and $\phi \gg M$ we find that indeed $m_\phi \gg H$ for cases.

For the former case of $\phi \lesssim M$, by equation (2.39)

$$\frac{m_\phi^2}{H^2} \gtrsim 3\beta\Omega_m n \frac{M}{M_{\rm Pl}} \tag{2.40}$$

However, during both the radiation dominated era and the matter dominated era Ω_m is monotonically increasing. That is, Ω_m is monotonically increasing with time, and so for all relevant times since the Hot Big Bang until today we have

$$\Omega_m \gtrsim 10^{-28}$$

Thus with $M \lesssim 10^{-3}$eV as given by the bound (2.17) for equation (2.40) we get

$$\frac{m_\phi^2}{H^2} > 3\beta n(10^2) \tag{2.41}$$

which, for both β and n of order one, is much larger than one.

On the other hand, if $\phi \gg M$ then equation (2.39) becomes

$$\frac{m_\phi^2}{H^2} \simeq 3\beta(n+1)\frac{M}{\phi_{\rm min}}\frac{M_{\rm Pl}}{M}\Omega_m \tag{2.42}$$

on ignoring the $\left(\frac{M}{\phi_{\rm min}}\right)^n$ term which goes to zero.

Furthermore, using approximation (2.22), equation (2.20) becomes

$$\left(\frac{M}{\phi_{\rm min}}\right)^{n+1} \simeq \frac{\beta}{n}\frac{M}{M_{\rm Pl}}\Omega_m\frac{3H^2M_{\rm Pl}^2}{M^4} \tag{2.43}$$

Then for equation (2.42) this yields

$$\frac{m_\phi^2}{H^2} \simeq 3(n+1)\beta\left(\frac{\beta}{n}\right)^{\frac{1}{n+1}}\left[\left(\frac{M_{\rm Pl}}{M}\right)^n\Omega_m^{n+2}\frac{3H^2M_{\rm Pl}^2}{M^4}\right]^{\frac{1}{n+1}}$$

or

$$\frac{m_\phi^2}{H^2} \simeq \left[\left(\frac{M_{\rm Pl}}{M}\right)^n\Omega_m^{n+2}\frac{3H^2M_{\rm Pl}^2}{M^4}\right]^{\frac{1}{n+1}} \tag{2.44}$$

on ignoring the constant term in β and n.

Here the $H^2M_{\rm Pl}^2$ term on the right-hand-side is decreasing with time, and with $M \lesssim 10^{-3}$eV as per the bound (2.17) is of the order of M^4 today. Hence for all relevant times since the Hot Big Bang until today we find that

$$\frac{m_\phi^2}{H^2} \simeq \left[\left(\frac{M_{\rm Pl}}{M}\right)^n\Omega_m^{n+2}\right]^{\frac{1}{n+1}} \tag{2.45}$$

We recall that by equation (2.28)

$$\Omega_m \simeq 10^{-6}$$

when $\phi_{\rm min} \simeq M$. Thus, since Ω_m is an increasing function with time in both the radiation-dominated era and the matter-dominated era we have

$$\Omega_m \gg 10^{-6}$$

when $\phi \gg M$. So using equation (2.17) we get

$$\frac{m_\phi^2}{H^2} \gg 10^{\frac{12(2n-1)}{n+1}} \tag{2.46}$$

Here the right-hand-side is greater than one for n greater than order one. Therefore we see that indeed $m_\phi > H$ when $\phi \gg M$.

So overall we find that the chameleon field follows an attractor solution given by the minimum of the effective potential such that

$$\phi(t) = \phi_{\rm min}(t) \tag{2.47}$$

for all relevant times since the Hot Big Bang until today.

Eventually the matter energy density ρ_m will become dominated by the vacuum energy density such that we may take

$$\rho_m \to 0$$

Thus the universe will evolve as a de Sitter universe. The chameleon field will follow the minimum of the effective potential energy function until $m_\phi \simeq H$, whence the chameleon field will no longer follow the minimum of the effective potential adiabatically. Rather it will lag behind the minimum of the effective potential. Therefore the field ϕ becomes the usual quintessence scalar field.

2.7 The Dynamics Along The Attractor

We next show that the chameleon field is slow rolling. In addition we show that the equation of state of the chameleon field is different from that of the usual quintessence theory, both of which are different from the

$$w_\Lambda = -1$$

equation of state of a cosmological constant. Furthermore, we find that the equation of state of the chameleon field need not be constant.

By equation (2.7) we have

$$-V_{,\phi}(\phi_{\rm min}) = \frac{\beta}{M_{\rm Pl}} \rho_m \exp\left(\frac{\beta\phi_{\rm min}}{M_{\rm Pl}}\right)$$

Thus by equation (2.25) we get

$$m^2 \simeq -V_{,\phi}(\phi_{\rm min}) \frac{\beta}{M_{\rm Pl}}$$

Then taking the time derivative gives

$$\dot{\phi}_{\text{min}} \simeq -3H\frac{V_{,\phi}(\phi_{\text{min}})}{V_{,\phi\phi}(\phi_{\text{min}})} \tag{2.48}$$

Thus, using equation (2.25) and the definition of Γ (2.10) we see that

$$\frac{\dot{\phi}^2_{\text{min}}(t)}{2V(\phi_{\text{min}}(t))} \simeq \frac{9}{2}\frac{H^2}{m^2_\phi}\frac{1}{\Gamma} \tag{2.49}$$

So since $m^2_\phi \gg H^2$ according to our discussion in section 2.6, and since we have demanded that $\Gamma > 1$ in section 2.1, we get

$$\frac{\dot{\phi}^2_{\text{min}}(t)}{2V\phi_{\text{min}}(t)} \ll 1 \tag{2.50}$$

Therefore the chameleon field is slow-rolling along the attractor solution that is the minimum of the effective potential energy function.

Since the chameleon field is non-minimally coupled we have

$$w \neq \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{2.51}$$

Rather, with the energy density of the chameleon field given by equation (2.32), through equation (2.50) we see that at the minimum

$$\rho_\phi \simeq V(\phi_{\text{min}})$$

Hence taking the time derivative of ρ_ϕ, the definition (2.10) and equation (2.48) yield

$$\begin{aligned}\frac{\dot{\rho}_\phi}{\rho_\phi} &\simeq \frac{V_{,\phi}(\phi_{\text{min}})}{V(\phi_{\text{min}})}\dot{\phi}_{\text{min}} \\ &= -3H\frac{1}{\Gamma}\end{aligned} \tag{2.52}$$

We recall that the continuity equation is given by

$$\frac{\dot{\rho}_\phi}{\rho_\phi} = -3H(1 + w_{\text{Eff}}) \tag{2.53}$$

Accordingly for the chameleon field we must have

$$w_{\text{Eff}} = \frac{1}{\Gamma} - 1 \tag{2.54}$$

Differentiating the fiducial potential (2.13) we get

$$\Gamma(\phi_{\text{min}}(t)) = 1 + \left(1 + \frac{1}{n}\right)\left(\frac{\phi_{\text{min}}(t)}{M}\right)^n \tag{2.55}$$

Thus by equation (2.54), if $\phi_{\text{min}} \ll M$ then $\Gamma \simeq 1$, giving

$$w_{\text{Eff}} \simeq 0$$

while if $\phi_{\text{min}} \gg M$ we get $\Gamma \gg 1$, giving

$$w_{\text{Eff}} \simeq -1$$

However, as per our discussion at the end of section 2.4, $\phi_{\text{min}} \simeq M$ at a temperature of $T_{\text{Crit}} \simeq 10\,\text{MeV}$, as in equation (2.27). Therefore

$$w_{\text{Eff}} \simeq 0$$

at early times in the universe when $T \gg 10\,\text{MeV}$, while

$$w_{\text{Eff}} \simeq -1$$

at late times in the universe when $T \ll 10\,\text{MeV}$. That is, in the early universe the chameleon field has equation of state similar to that of matter, while in the late universe it has equation of state similar to a cosmological constant.

Usually the equation of state is given by equation (2.51), which reduces to

$$\begin{aligned} w_{\text{usual}} &= \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \\ &= \frac{\frac{\dot{\phi}^2}{2V(\phi)} - 1}{1 + \ldots} \\ &\simeq \frac{\dot{\phi}^2}{2V(\phi)} - 1 \end{aligned} \tag{2.56}$$

by equation (2.50). Equivalently, by equation (2.49) we see that

$$w_{\text{usual}} \simeq \frac{9}{2}\frac{H^2}{m^2}\frac{1}{\Gamma} - 1$$

For a quintessence theory the equation of state is given by

$$w_Q \simeq \frac{w_B - 2(\Gamma - 1)}{1 + 2(\Gamma - 1)} \tag{2.57}$$

where w_B is the equation of state for a perfect fluid background. By contrast, for the chameleon field the effective equation of state is independent of the background equation of state. Consequently, the evolution of the chameleon field is insensitive to whether the universe is in a radiation dominated era or a matter dominated era.

2.8 Approaching The Attractor

Moving on, during the course of the next three sections, we consider three cases for the initial value of the chameleon field. Explicitly, we consider the case of the initial field value being at the minimum of the effective potential energy function, that of the initial field value being much larger than the minimum, and finally that of the initial field value being much smaller than the minimum.

We assume that initially the chameleon field has vanishing kinetic energy. In addition, we let both $\rho_\phi < \rho_m$ and $\rho_\phi < \rho_r$ initially, as per the case of equipartition at reheating. For some initial value of the chameleon field ϕ_i, say, at time t_i, say, we recall that the effective potential has a minimum for some $\phi_{\text{min}}(t_i)$, say. If

$$\phi_i = \phi_{\text{min}}(t_i)$$

then the chameleon field will follow this minimum. Similarly, as long as it is neither the case that $\phi_i \gg \phi_{\text{min}}(t_i)$ nor $\phi_i \ll \phi_{\text{min}}(t_i)$ then the chameleon field will undergo small oscillations about the minimum, eventually settling to the minimum $\phi_{\text{min}}(t_i)$ of the effective potential.

2.9 Undershooting

Next we consider the non-trivial case in which the initial value of the chameleon field is much larger than the minimum of the effective potential energy function such that

$$\phi_i \gg \phi_{\text{min}}(t_i)$$

In this case we may neglect the $V_{,\phi}(\phi)$ term in equation (2.31). We reintroduce the stress-energy-momentum tensor $T^{\mu}{}_{\mu}$ whence the Klein-Gordon equation of motion becomes

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\beta}{M_{\rm Pl}} T^{\mu}{}_{\mu} \tag{2.58}$$

Here the $\dot{\phi}$ term due to the expansion of the universe acts like a friction term, while the $T^{\mu}{}_{\mu}$ term acts like a force term.

If $T^{\mu}{}_{\mu}$ is negligibly small for relativistic degrees of freedom then we may put

$$T^{\mu}{}_{\mu} \simeq -\rho_m \tag{2.59}$$

However, in this case, during the radiation dominated era we get

$$\rho_m \ll 3H\dot{\phi}$$

and so we find that the chameleon field is over-damped. Then the chameleon field would remain frozen at its initial value ϕ_i. That is, it would not be at the minimum of the effective potential.

During the evolution of the early universe particles of species i become non-relativistic when the temperature of the universe satisfies

$$T \simeq m_i$$

where m_i is the mass of the particle of species i. This gives a non-zero trace of the stress-energy-momentum tensor for about one e-fold of expansion. Hence the chameleon field is driven towards the minimum of the effective potential $\phi_{\rm min}$. Namely, each particle species contributes

$$T^{\mu(i)}_{\mu} = -\frac{45}{\pi} H^2 M_{\rm Pl}^2 \frac{g_i}{g_*(T)} \tau\left(\frac{m_i}{T}\right) \tag{2.60}$$

where

$$g_*(T) = \sum_{\rm Bosons} g_i^{\rm (Boson)} \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\rm Fermions} g_i^{\rm (Fermion)} \left(\frac{T_i}{T}\right)^4 \tag{2.61}$$

is the effective number of relativistic degrees of freedom, while the g_i are the number of relativistic degrees of freedom. Furthermore, τ is given by

$$\tau(x) = x^2 \int_x^{\infty} \frac{\sqrt{u^2 - x^2}}{\exp(u) \pm 1}\, {\rm d}u \tag{2.62}$$

with a plus sign for Fermions, and a minus sign for Bosons, where

$$x = \frac{m_i}{T}$$

Here $\tau(x) \ll 1$ for both $x \ll 1$ and $x \gg 1$, while $\tau(x) \simeq 1$ for $x \simeq 1$.

From the equation of motion (2.58), numerical calculations[29] show that the value of the chameleon field shifts from its initial value by an amount

$$(\Delta\phi_i) = -\beta \frac{g_i}{g_*(m_i)} M_{\rm Pl} \times A \tag{2.63}$$

with $A = \frac{7}{8}$ for Fermionic particle species and $A = 1$ for Bosonic particle species. We here omit the full calculation which is carried out in Appendix **B** of [29]

The total change in the value of the chameleon field is given by summing the contribution to $\Delta\phi$ due to each relevant particle species. We must have the chameleon field at the minimum of the effective potential energy function by the start of Big Bang Nucleosynthesis. This occurs at a temperature of $T \simeq 1\,\text{MeV}$. Hence the electron and positron do not contribute to pushing the chameleon field towards the minimum value $\phi_{\rm min}$, since they do not become non-relativistic before the start of Big Bang Nucleosynthesis. Thus we get a total change of

$$(\Delta\phi_i)_{\rm Total} \simeq -\beta M_{\rm Pl} \tag{2.64}$$

Therefore, if $\phi_i \lesssim M_{\rm Pl}$ the value of the chameleon field will be driven towards the minimum of the effective potential. The chameleon field then undergoes oscillations until it settles to the minimum of the effective potential. If it reaches the minimum of the effective potential before the final relevant particle has become non-relativistic, the kick due to that and any subsequent particle simply causes the chameleon field to start oscillating about the minimum of the effective potential. It then oscillates until it again settles to the minimum value.

2.10 Overshooting

Finally we consider the case of

$$\phi_i \ll \phi_{\rm min}(t_i)$$

such that the initial value of the chameleon field is much less than the minimum of the effective potential energy function. Here the matter density term ρ_m is negligible.

Hence the Klein-Gordon equation (2.31) becomes

$$\ddot{\phi} + 3H\dot{\phi} = -V_{,\phi}(\phi) \tag{2.65}$$

and this is a minimally coupled scalar field.

However, for $\phi_i \ll \phi_{\rm min}(t_i)$ by equation (2.8) we find that

$$V_{,\phi\phi}(\phi_i) \gg m_i^2$$

where

$$m_i^2 = V_{,\phi\phi}\left(\phi_{\rm min}(t_i)\right)$$

Then since $m_\phi^2 \gg H^2$ as per our discussion in section 2.6 for all relevant times since the Hot Big Bang until today we find that

$$V_{,\phi\phi}(\phi_i) \gg H_i^2$$

where $H_i = H(t_i)$ is the value of the Hubble parameter initially. Hence the chameleon field is under-damped. Therefore the dynamics of the chameleon field are dominated by the $V_{,\phi}(\phi)$ term in the field equation of motion, and so it behaves like a free field.

Thus most of the energy of the chameleon field is in the form of kinetic energy and it becomes kinetic energy dominated such that

$$\dot{\phi}^2 \gg V(\phi)$$

However we have $\phi_i \ll \phi_{\rm min}(t_i)$, and so this implies that the chameleon field will roll past the minimum value of the effective potential. That is, it will *Overshoot* the minimum. As it rolls the energy of the chameleon field gets red-shifted, which means that the damping term becomes important. Eventually the chameleon field stops for some value

$$\phi_{\rm stop} > \phi_{\rm min}$$

Since we know that $\dot{\phi} \propto a^{-3}(t)$ and that $a \propto t^{\frac{1}{2}}$ during the radiation dominated era we find that

$$a\frac{\mathrm{d}\phi}{\mathrm{d}a} = \sqrt{6\Omega_\phi^{(i)}}\left(\frac{a_i}{a}\right) M_{\rm Pl} \tag{2.66}$$

where $a_i = a(t_i)$. Then integrating from

$$\phi_i = \phi(t_i) \qquad \text{and} \qquad a_i = a(t_i)$$

yields

$$\phi(a) = \phi_i + \sqrt{6\Omega_\phi^{(i)}}\, M_{\rm Pl} \left(1 - \frac{a_i}{a}\right) \tag{2.67}$$

In the limit of $a \gg a_i$ this becomes

$$\phi(a) = \phi_i + \sqrt{6\Omega_\phi^{(i)}}\, M_{\rm Pl} \tag{2.68}$$

That is, we get the undershooting case, but now we have the initial undershooting field value $\phi_i^{(\rm undershoot)}$ given by the value at which the field stopped rolling, namely $\phi_{\rm stop}$.

2.11 Converging To The Minimum

Continuing onwards, we study how the chameleon field settles to the minimum of the effective potential when it has a field value near to but different from that minimum.

We assume that a linear approximation is sufficient to describe the effective potential energy function of the chameleon field, such that we get harmonic oscillations. Moreover, we neglect the kinetic energy term from the final particle contribution to the change in the chameleon field value. Then, in this linear approximation our effective potential becomes

$$V_{\rm Eff}(\phi) \simeq \frac{1}{2} m_\phi^2(t) \left[\phi(t) - \phi_{\rm min}(t)\right]^2 \tag{2.69}$$

Thus the total energy is

$$\rho_\phi \simeq \frac{1}{2} m_\phi^2(t) \left[\dot\phi(t) - \dot\phi_{\rm min}(t)\right]^2 + \frac{1}{2} m_\phi^2(t) \left[\phi(t) - \phi_{\rm min}(t)\right]^2 \tag{2.70}$$

So averaging over many oscillations gives

$$\rho_\phi \simeq \frac{1}{2} m_\phi^2(t) \left\langle \dot\phi(t) - \dot\phi_{\rm min}(t) \right\rangle^2 \tag{2.71}$$

Now, since $m \gg H$, as discussed in section 2.6, this means that this is analogous to an oscillating pendulum which is slowly being lengthened[31]. Hence we get a conserved quantity given by

$$N = \frac{\rho_\phi \phi^3}{m_\phi(t)} \tag{2.72}$$

Accordingly,

$$m_\phi(t) \left\langle \dot{\phi}(t) - \dot{\phi}_{\min}(t) \right\rangle^2 \propto a^{-3} \tag{2.73}$$

However since $m_\phi(t)$ is a function of time, and since by equation (2.8)

$$m_\phi(t) \simeq \sqrt{V_{,\phi\phi}\left(\phi_{\min}(t)\right)}$$

by substituting $\dot{\phi}_{\min}$ as given in equation (2.48) we see that

$$\frac{\dot{m}_\phi(t)}{m_\phi(t)} \simeq \frac{3}{2} H \frac{V_{,\phi\phi\phi}(\phi) V_{,\phi}(\phi)}{V^2_{,\phi\phi}(\phi)} \tag{2.74}$$

For $V(\phi) \propto \phi^{-n}$ as in equation (2.18) we get

$$\begin{aligned} \frac{\dot{m}_\phi(t)}{m_\phi(t)} &\simeq \frac{3}{2} H \frac{(n+2)(n+1)}{(n+1)^2} \\ &\simeq \frac{3}{2} H \frac{(n+2)}{(n+1)} \end{aligned} \tag{2.75}$$

Consequently, integrating equation (2.75), with

$$H = \frac{\dot{a}(t)}{a(t)}$$

we find that

$$m_\phi(t) \propto a^{\frac{-3(n+2)}{2(n+1)}} \tag{2.76}$$

So, by equation (2.73), we find that

$$\left\langle \dot{\phi}(t) - \dot{\phi}_{\min}(t) \right\rangle^{\frac{1}{2}} \propto a^{\frac{-3n}{4(n+1)}} \tag{2.77}$$

2.12 Constraints On Initial Conditions

We now wish to explore the bound on the initial energy density of the chameleon field that is set by the experimental constraint on the change in the mass of the elementary particles since the Big Bang Nucleosynthesis until today. From our previous discussion of the undershooting case of the initial value of the chameleon field in section 2.9 we already have an upper bound on the initial value of the chameleon field given by equation (2.64).

By the conformal coupling (2.3) of the chameleon field to the matter fields, a constant mass $m^{(i)}$ in the matter-frame is related to a mass

$$m = m(\phi)$$

in the Einstein frame by the re-scaling

$$m(\phi) = \exp\left(\frac{\beta_i \phi}{M_{\rm Pl}}\right) m^{(i)}$$

Thus performing a variational calculation we see that a variation in the chameleon field $\Delta\phi$ causes a variation in the mass of the matter fields Δm of

$$\left|\frac{\Delta m(\phi)}{m(\phi)}\right| \simeq \frac{\beta}{M_{\rm Pl}} |\Delta\phi| \tag{2.78}$$

Measurements of the abundance of light elements constrain the variation in the mass $\Delta m(\phi)$ between the onset of Big Bang Nucleosynthesis, $t_{\rm Nuc}$, and today, t_0, to be less than about 10%. Thus we need

$$\left|\phi_{\rm BBN} - \phi^{(0)}\right| \lesssim 0.10 \times \frac{M_{\rm Pl}}{\beta}$$

where $\phi_{\rm BBN}$ is the value of the chameleon field at the start of Big Bang Nucleosynthesis and $\phi^{(0)}$ is the value of the chameleon field today. Since today we have $\phi_{\rm min} \ll M_{\rm Pl}$ this means that the chameleon field must satisfy

$$\phi_{\rm BBN} \lesssim 0.10 \times \frac{M_{\rm Pl}}{\beta} \tag{2.79}$$

Given that we require $\phi_i \lesssim \beta M_{\rm Pl}$ and $\phi_{\rm stop} \lesssim \beta M_{\rm Pl}$, on neglecting the initial field value ϕ_i and letting β be of order one, equation (2.68) gives the bound

$$\Omega_\phi^{(i)} \lesssim \frac{1}{6} \tag{2.80}$$

2.13 The Chameleon Field During Inflation

It remains to examine the behaviour of a universe with a chameleon field which undergoes very early universe cosmic inflation.

During the inflationary epoch our effective potential energy function (2.6) becomes[29]

$$V_{\text{Eff}}(\phi) = M^4 \exp\left(\frac{M^n}{\phi^n}\right) + \rho_{\text{Vac}} \exp\left(\frac{4\beta\phi}{M_{\text{Pl}}}\right) \tag{2.81}$$

where ρ_{Vac} is the vacuum energy of the inflaton field. So in analogy with the analysis in section 2.4, this effective potential (2.81) has a minimum for some value $\phi_{\min}$ of the chameleon field which satisfies $\phi_{\min} \ll M$. However ρ_{Vac} is constant, and so the minimum of the effective potential is constant during the inflationary epoch. Therefore the chameleon field is stable.

In analogy to the derivation of equation (2.39), using the effective potential (2.81) we find that the mass of the chameleon field at the minimum of the effective potential satisfies

$$m_\phi^2 \simeq 12\beta n \frac{M_{\text{Pl}}}{M} \left(\frac{M}{\phi_{\min}}\right)^{n+1} H^2 \tag{2.82}$$

and so again we see that $m_\phi^2 \gg H^2$. However, m_ϕ is constant during the inflationary epoch, and so the chameleon field oscillates about the minimum of the effective potential with amplitude

$$\left\langle (\phi(t) - \phi_{\min}(t))^2 \right\rangle \propto a^{-3} \tag{2.83}$$

Hence the chameleon field behaves like dust matter during the period of cosmic inflation. However, during the inflationary epoch the scale factor a grows exponentially, and so the chameleon field settles to the minimum of the effective potential rapidly.

During the reheating phase of the very early universe production of quanta of the chameleon field is suppressed due to its large mass. Thus the universe becomes radiation dominated at the end of the cosmic inflationary epoch. Hence the minimum of the effective potential shifts to a very large value of the chameleon field, such that at the end of the cosmic inflationary epoch

$$\phi_i \ll \phi_{\min}$$

Therefore we find that an overshooting scenario is the likely outcome given the occurrence of a cosmic inflationary epoch.

Chapter 3

The Cosmic Microwave Background

3.1 The Cosmic Microwave Background

For this second part of the essay we wish to consider the modification of the cosmic microwave background anisotropies due to the presence of the chameleon field. In doing so we follow [32] closely throughout. Moreover, we cite this paper and the references therein on many occasions.

This modification to the cosmic microwave background anisotropies occurs due to interactions between the chameleon field and the photon field in the presence of a magnetic field. Hence such interactions may occur in the intra-cluster-medium of galaxy clusters. To investigate this effect we must first discuss the cosmic microwave background. Following this we give an introduction to the theory of the magnetic fields of galaxy clusters. We focus on two models, namely the cell model and the power spectrum model. Next we discuss consequences of the standard thermal Sunyaev-Zel'Dovich effect on the cosmic microwave background. We will then be able to examine the modified chameleonic Sunyaev-Zel'Dovich Effect.

To begin, we recall that the cosmic microwave background is approximately uniform across the sky to one part in ten-thousand, with a black-body intensity spectrum given by

$$I_0 = \frac{k_B T_0}{2\pi^2}\frac{\omega^2}{c^2}\frac{x}{\exp(x)-1} \tag{3.1}$$

with a temperature of

$$T_0 = 2.725\text{K}$$

where

$$x = \frac{\omega}{k_B T}$$

with ω the frequency and $k_B = 1.38 \times 10^{23}\mathrm{JK}^{-1}$ is the usual Boltzmann constant.

Small fluctuations in the cosmic microwave background are due to primordial density fluctuations at the time of emission of the cosmic microwave background radiation some $380,000$ years after the Big Bang. These are related to changes in the black-body temperature by

$$\frac{\delta I}{I_0} = \frac{x}{1-\exp(-x)}\frac{\delta T}{T_0} \tag{3.2}$$

The root-mean-square average of these primary fluctuations is about 100μK.

We now wish to consider the possibility for an interaction between the chameleon field and the electromagnetic field. We extend the scalar-tensor action (2.1) for the chameleon field so as to include the electromagnetic field tensor $\mathcal{F}^{\mu\nu}$ such that

$$S \to S_I = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{M_{\mathrm{Pl}}^2}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{4}\frac{\phi}{M_F}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right) - \int \mathrm{d}^4x\,\mathcal{L}_m(\psi_m^{(i)}, g_{\mu\nu}^{(i)}) \tag{3.3}$$

where as usual

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

and M_F^{-1} is the coupling strength of the electromagnetic interaction between the chameleon field and the photon field. Here the coupling of the chameleon field to the matter fields is in general not the same as the coupling of the chameleon field to the photon field. This interaction alters the intensity and polarisation of radiation which enters a magnetic region. Therefore the observational effects may be detected in the laboratory or in some astro-physical phenomenon.

By performing a variational calculation we get an equation of motion for the chameleon field given by

$$\Box\phi = \frac{\partial}{\partial\phi}\left[V(\phi) + \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{4}\frac{\phi}{M_F}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + T_m\right] \tag{3.4}$$

where

$$T_m = \rho_m - 3P_m$$

is the trace of the stress-energy-momentum tensor. We see that this equation of motion is non-linear in the chameleon field ϕ. In particular, we recall that the mass of the chameleon field is heavier in dense environments, as per the discussion in section 2.1. We also recall that for small perturbations about the minimum of the effective potential energy function the mass of the chameleon field is given by equation (2.8).

In order to conform to the bound set by the PVLAS experiment we must have $M_F \gtrsim 2\times 10^6$ GeV[33, 34]. In addition, the CAST experiment at CERN[35] sets a bound of $M_F \gtrsim 1.1\times 10^9$ GeV[36].

We recall that for the fiducial potential (2.13), with $M \lesssim 10^{-3}$eV as given by the bound (2.17), expanding gives (2.18)

$$V(\phi) \simeq M^4 + M^4\left(\frac{M^n}{\phi^n}\right) + \dots$$

That is, we get a constant term $M^4 \simeq 10^{-12}$eV of the same order as the vacuum energy density today. This gives a mass term for the chameleon field of

$$m_\phi \ll 10^{-14}\text{eV}$$

in the galactic and intra-cluster-medium.

We consider the case of the cosmic microwave background radiation propagating through the magnetic field of a galaxy cluster. Then interactions between the cosmic microwave background photons and the chameleon field occur. Therefore, the cosmic microwave background intensity and polarisation become altered. Again by performing a variational calculation we get an equation of motion for the electromagnetic field given by

$$\nabla_\mu\left[\mathcal{F}^{\mu\nu} + \frac{\phi}{M_F}\mathcal{F}^{\mu\nu}\right] = J^\nu \tag{3.5}$$

where J^ν is a conserved current such that

$$\nabla_\mu J^\mu = 0$$

We consider a sparse astro-physical background with a magnetic field $\vec{B}(x^\rho)$. For electromagnetic radiation of frequency ω the chameleon field and the photon field vary over a timescale of ω^{-1}, that is, slowly. We let perturbations in the chameleon field ϕ and the photon field a_μ be given by

$$\phi = \Phi - \bar{\Phi} \qquad \text{and} \qquad a_\mu = A_\mu - \bar{A}_\mu$$

respectively.

The plasma frequency in the intra-cluster-medium is given by[32]

$$\omega_{\rm Pl}^2 = \frac{4\pi\alpha_{\rm EM} n_e}{m_e}$$

where n_e is the electron number density and m_e is the mass of the electron. Then if $\omega \gg H$, ignoring the second order perturbation terms gives equations of motion

$$-\ddot{\vec{a}} + \nabla^2\vec{a} \simeq \frac{1}{M_F}\vec{\nabla}\phi\times\vec{B} + \omega_P^2\vec{a} \tag{3.6}$$

and

$$-\ddot{\phi} + \nabla^2\phi \simeq \frac{1}{M_F}\vec{B}\cdot(\vec{\nabla}\times\phi) + m_\phi^2\phi \tag{3.7}$$

respectively.

We consider photons propagating in the z-direction such that

$$\vec{a} = \begin{pmatrix} \gamma_x(z) \\ \gamma_y(z) \\ 0 \end{pmatrix}$$

We aim to obtain an equation of motion for the chameleon-photon system. We follow an approach similar to (15 in [32]), however, here we simply quote the results. To start, we make a redefinition

$$\gamma_i(z) = \tilde{\gamma}(z)_i \exp[i\omega(z-t) + i\beta(z)] \qquad \text{and} \qquad \phi(z) = \tilde{\phi}(z)\exp[i\omega(z-t) + i\beta(z)]$$

where

$$\frac{\partial}{\partial z}\beta(z) = -\frac{\omega_{\rm Pl}^2(z)}{2\omega}$$

Then approximating

$$-\partial_t^2 \simeq \omega^2 \qquad \text{and} \qquad \omega^2 + \partial_z^2 \simeq 2\omega(\omega + i\partial_z)$$

and making the redefinition

$$\vec{u} = \begin{pmatrix} \tilde{\gamma}_x(z) \\ \tilde{\gamma}_y(z) \\ \exp\left[2i\Delta(z)\tilde{\phi}(z)\right] \end{pmatrix}$$

where

$$\Delta(z) = \int_0^z \frac{m_{\mathrm{Eff}}^2(x)}{4\omega} \mathrm{d}x \tag{3.8}$$

with

$$m_{\mathrm{Eff}}^2 = m_\phi^2(z) - \omega_{\mathrm{Pl}}^2(z) \tag{3.9}$$

we get an equation of motion[32] for both ϕ and $\vec{a}$ given by

$$\frac{\partial}{\partial z}\vec{u} = \frac{\mathcal{B}(z)}{2M_F}\vec{u} \tag{3.10}$$

where

$$\mathcal{B}(z) = \begin{pmatrix} 0 & 0 & -B_y \exp[-2i\Delta(z)] \\ 0 & 0 & B_x \exp[-2i\Delta(z)] \\ B_y \exp[2i\Delta(z)] & -B_x \exp[2i\Delta(z)] & 0 \end{pmatrix}$$

We recall the *Stokes' Parameters* for the polarisation of photon radiation of intensity

$$I_\gamma(z) = |\gamma_x(z)|^2 + |\gamma_y(z)|^2 \tag{3.11}$$

linear polarisation

$$Q(z) = |\gamma_x(z)|^2 - |\gamma_y(z)|^2 \quad \text{and} \quad U(z) = 2\mathcal{R}\{\gamma_x(z)\gamma_y(z)\} \tag{3.12}$$

and circular polarisation

$$V(z) = 2\mathcal{I}\{\gamma_x^*(z)\gamma_y(z)\} \tag{3.13}$$

These will prove useful in describing the magnitude of the mixing between the chameleon field and the photon field. So, for a vanishing initial chameleon field flux

$$\begin{aligned} I_\phi &= |\phi|^2 \\ &= 0 \end{aligned}$$

the final photon intensity is given by[32]

$$I_\gamma(z) = I_\gamma(0)\left[1 - \mathcal{P}_{\gamma\leftrightarrow\phi}(z)\right] + Q(0)\mathcal{Q}_q(z) + U(0)\mathcal{Q}_u(z) + V(0)\mathcal{Q}_v(z) \tag{3.14}$$

where

$$\mathcal{P}_{\gamma\leftrightarrow\phi}(z) = \frac{1}{2}\left[|A_x(z)|^2 + |A_y(z)|^2\right] \tag{3.15}$$

while

$$\mathcal{Q}_q(z) = \frac{1}{2}\left[|A_x(z)|^2 - |A_y(z)|^2\right] \tag{3.16}$$

and

$$\mathcal{Q}_u(z) = \mathcal{R}\{A_x^*(z)A_y(z)\} \tag{3.17}$$

with

$$\mathcal{Q}_v(z) = \mathcal{I}\{A_x^*(z)\gamma_y(z)\} \tag{3.18}$$

where

$$A_i(z) = \int_0^z \frac{B_i(x)\exp[2i\Delta(x)]}{2M_F}\,\mathrm{d}x$$

In addition, we recall that the cosmic microwave background has a very small intrinsic polarisation, with a linear polarisation of

$$\langle Q(z)\rangle^{\frac{1}{2}} \simeq 10^{-6} \qquad \text{and} \qquad \langle U(z)\rangle^{\frac{1}{2}} \simeq 10^{-6} \tag{3.19}$$

and a vanishing circular polarisation

$$V(z) = 0 \tag{3.20}$$

such that

$$\frac{1}{I_\gamma(0)}\sqrt{Q^2(0) + U^2(0) + V^2(0)} \ll 1$$

Therefore the Stokes' parameters for the modification of the cosmic microwave background intensity and polarisation due to the interactions between the chameleon field and the photon field are

$$\begin{aligned} \Delta\vec{S}(z) &= \Delta\begin{pmatrix} I_\gamma(z) \\ Q(z) \\ U(z) \\ V(z)\end{pmatrix} \\ &\simeq \begin{pmatrix} -\mathcal{P}_{\gamma\leftrightarrow\phi}(z) \\ \mathcal{Q}_q(z) \\ \mathcal{Q}_u(z) \\ \mathcal{Q}_v(z)\end{pmatrix} I_\gamma(0) \end{aligned} \tag{3.21}$$

A typical value for the electron number density for a galaxy cluster is

$$10^{-3}\text{cm} \lesssim n_e \lesssim 10^{-2}\text{cm} \tag{3.22}$$

giving a plasma frequency of $\omega_{\text{Pl}} \simeq 10^{-12}$eV. Thus for a chameleon field the plasma frequency satisfies

$$\omega_{\text{Pl}}^2 \gg m_\phi^2$$

Henceforth we assume that the chameleon field is slim such that equation (3.9) becomes

$$m_{\text{Eff}}^2 \simeq -\omega_{\text{Pl}}^2$$

with $\omega_{\text{Pl}}^2 \propto n_e$. Therefore spatial variations in m_{Eff}^2 are due to variations in the electron number density n_e.

Finally, we recall that the *Sunyaev-Zel'Dovich Effect*[37, 38] causes warm spots in the cosmic microwave background intensity in the direction of galaxy clusters. This is because as cosmic microwave background photons pass through a galaxy cluster inverse Compton scattering with the electrons in the hot plasma core of the cluster cause a redistribution of the energy of the photons. That is, it causes a distortion in the frequency spectrum of the intensity. Hence there is a change in the apparent brightness of the cosmic microwave background radiation.

If the chameleon field couples to the photon field then there will be an additional alteration to the flux of the cosmic microwave background photons due to these interactions. Since the properties of the chameleon field are only weakly constrained in high density regions we may overcome the bounds mentioned earlier in this section due to the CAST and PVLAS experiments, for example. That is, we may have a *Chameleonic Sunyaev-Zel'Dovich Effect.*

Hence, measurements of the Sunyaev-Zel'Dovich effect may be compared to the modification made to the cosmic microwave background as predicted using a chameleon field theory. This would allow us to constrain the parameters of the chameleon field theory.

3.2 The Cell Model

The magnetic field of a galaxy may be modelled as having a regular part $\vec{B}_{\text{Reg}}(z)$ and a fluctuating part $\delta\vec{B}(z)$. This fluctuating magnetic field has a first-order variation and changes its direction on small scales. In the *Cell Model* we let these changes of direction occur on a scale of L_{Coh}. Then we say that a path of length L, say, crosses N magnetic domains, where

$$N = \frac{L}{L_{\text{Coh}}}$$

Thus the magnetic field strength and the electron number density are constant along the path, but the fluctuating magnetic field component $\delta\vec{B}(z)$ points in a random direction in each of the N domains.

We wish to calculate the magnitude of the mixing between the chameleon field and the photon field. As we have been doing, we continue to follow [32] in our calculations. We let

$$\vec{B}(z) = \vec{B}_{\text{Reg}} + \delta\vec{B}(z)$$

where $\vec{B}_{\text{Reg}}$ is constant over the path from the distant galaxy to the Earth-bound observer. In addition, we let B_{Rand} be defined by

$$(\delta\vec{B})_x = B_{\text{Rand}}\cos(\theta_n) \qquad \text{and} \qquad (\delta\vec{B})_y = B_{\text{Rand}}\sin(\theta_n)$$

for $(n-1)L_{\text{Coh}} < z < nL_{\text{Coh}}$ where the $\theta_n \in (0, 2\pi]$ are independent random variables. We define $\bar{\Delta}$ by

$$\bar{\Delta} = \frac{m_{\text{Eff}}^2 L}{4\omega} \tag{3.23}$$

Then, to quote [32], the expectation values of the parameters (3.15), (3.16), (3.17) and (3.18) are

$$\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}(z) = \frac{1}{2}\left(\frac{2\vec{B}_{\text{Reg}}\omega}{M_F m_{\text{Eff}}^2}\right)^2 \sin^2(\bar{\Delta}) + \frac{1}{2}N\left(\frac{2B_{\text{Rand}}\omega}{M_F m_{\text{Eff}}^2}\right)^2 \sin^2\left(\frac{\bar{\Delta}}{N}\right) \tag{3.24}$$

with

$$\bar{\mathcal{Q}}_q(z) = \frac{1}{2}\cos(2\theta_{\text{Reg}})\left(\frac{2\vec{B}_{\text{Reg}}\omega}{M_F m_{\text{Eff}}^2}\right)^2 \sin^2(\bar{\Delta}) \tag{3.25}$$

and

$$\bar{\mathcal{Q}}_u(z) = \frac{1}{2}\cos(2\theta_{\text{Reg}})\left(\frac{2\vec{B}_{\text{Reg}}\omega}{M_F m_{\text{Eff}}^2}\right)^2 \tag{3.26}$$

while

$$\bar{\mathcal{Q}}_v(z) = 0 \tag{3.27}$$

where $\delta\vec{B}$ has zero expectation value

$$\left\langle \delta\vec{B} \right\rangle = 0$$

and the θ_{Reg} are defined by

$$(\vec{B}_{\mathrm{Reg}})_x = B_{\mathrm{Reg}} \cos(\theta_{\mathrm{Reg}}) \qquad \text{and} \qquad (\vec{B}_{\mathrm{Reg}})_y = B_{\mathrm{Reg}} \sin(\theta_{\mathrm{Reg}})$$

Then if

$$\frac{\bar{\Delta}}{N} \gg 1 \tag{3.28}$$

when $N \gg 1$ we find that the variance of the $\mathcal{Q}_i$ is of the same order as $\bar{\mathcal{P}}^2_{\gamma\leftrightarrow\phi}$. If $B_{\mathrm{Rand}} \simeq B_{\mathrm{Reg}}$ then we see that many magnetic regions of coherence length $L_{\mathrm{Coh}} \ll L$ give a chameleon-photon mixing probability enhancement factor of about N.

3.3 The Power Spectrum Model

The cell model described above is not[32], however, accurate for regions in which $|\Delta(z = L)| \gg 1$, where $\Delta(z)$ is defined by equation (3.8). Furthermore, the cell model does not give a divergenceless fluctuating magnetic field (17 in [32]), such that

$$\vec{\nabla} \cdot \delta\vec{B} \neq 0$$

Rather, it is more accurate to consider a *Power Spectrum Model* in which the spectra of fluctuations from large scales to small scales have a correlation function given by

$$R_B(\vec{x}) = \left\langle \delta\vec{B}(\vec{y})\, \delta\vec{B}(\vec{x} + \vec{y}) \right\rangle$$

We let $\delta\vec{B}_i(\vec{x})$ be a Gaussian distributed random variable. In addition we let the fluctuations be isotropic. Then

$$\begin{aligned} (R_B)_{ij}(\vec{x}, \vec{y}) &= \left\langle \delta\vec{B}_i(\vec{y})\, \delta\vec{B}_j(\vec{x} + \vec{y}) \right\rangle \\ &= \frac{1}{3} R_B(\vec{x}, \vec{y}) \delta_{ij} \end{aligned} \tag{3.29}$$

where δ_{ij} is the usual Kronecker-delta. If the fluctuations are position independent we find that

$$R_B(\vec{x}, \vec{y}) \simeq R_B(x)$$

where $R_B(x)$ is the magnetic field auto-correlation function.

We let the electron number density be given by

$$n_e = \bar{n}_e + \delta n_e$$

Moreover, we let $\left(1 + \frac{\delta n_e}{n_e}\right)$ be a log-normally distributed random variable with mean 1 and variance $\langle \delta_n^2 \rangle$, where

$$\delta_n = \frac{\delta n_e}{\bar{n}_e}$$

Hence we define the electron number density auto-correlation function by

$$R_N(x) = \langle \delta n_e(\vec{y})\, \delta n_e(\vec{x} + \vec{y}) \rangle$$

The power spectra for the magnetic field and electron number density fluctuations are defined by

$$\begin{aligned} R_B(x) &= \frac{1}{4\pi} \int \mathrm{d}^3k \, \exp\left[2\pi i \vec{k} \cdot \vec{x}\right] P_B(k) \\ &= \int k^2 P_B(k) \frac{\sin(2\pi k x)}{2\pi k x} \, \mathrm{d}k \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} R_N(x) &= \frac{1}{4\pi} \int \mathrm{d}^3k \, \exp\left[2\pi i \vec{k} \cdot \vec{x}\right] P_N(k) \\ &= \int k^2 P_N(k) \frac{\sin(2\pi k x)}{2\pi k x} \, \mathrm{d}k \end{aligned} \tag{3.31}$$

Following (18 in [32]), we define the correlation length scale for the fluctuations by

$$L_B = \frac{\int_0^\infty k P_B(k) \, \mathrm{d}k}{2 \int_0^\infty k^2 P_B(k) \, \mathrm{d}k} \tag{3.32}$$

and

$$L_N = \frac{\int_0^\infty k P_N(k) \, \mathrm{d}k}{2 \int_0^\infty k^2 P_N(k) \, \mathrm{d}k} \tag{3.33}$$

We let $\bar{\Delta}$ as given by (3.23) be

$$\bar{\Delta} \simeq -\frac{4\pi \alpha_{\mathrm{EM}} \bar{n}_e L}{4 m_e \omega}$$

and let

$$k_{\text{Crit}} = \left|\frac{\bar{\Delta}}{\pi L}\right|$$

Furthermore we let there exist some $k_B^{(-3)} \ll k_{\text{Crit}}$ and some $k_N^{(-3)} \ll k_{\text{Crit}}$ such that for $k > k_B^{(-3)}$ and $k > k_N^{(-3)}$ each of $P_B(k)$ and $P_N(k)$ respectively decrease with k faster that k^{-3} as k approaches infinity. Accordingly the main contributions to $\left\langle \delta\vec{B}^2 \right\rangle$ and $\left\langle n_e^2 \right\rangle$ come from spatial scales larger than k_{Crit}^{-1}. Therefore we assume that $k_{\text{Crit}}^{-1} \ll L_B$ and $k_{\text{Crit}}^{-1} \ll L_N$. This means that

$$\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi} \simeq \frac{1}{2}\left[\frac{2B_{\text{Eff}}\omega}{M_F \bar{m}_{\text{Eff}}^2}\right]^2 I_N^3 - \frac{1}{4}\left[\frac{2\vec{B}_{\text{Reg}}\omega}{M_F \bar{m}_{\text{Eff}}^2}\right]^2 \cos\left(2\bar{\Delta}\right) + \frac{B_{\text{Eff}}^2 L}{8M_F^2 \bar{n}_e^2} I_N^2 W_N(k_{\text{Crit}}) + \frac{L}{24M_F^2} I_N^3 W_B(k_{\text{Crit}}) \quad (3.34)$$

where

$$I_N = 1 + \left\langle \delta_n^2 \right\rangle$$

and

$$B_{\text{Eff}}^2 = \frac{1}{2}\left\langle (\hat{z} \times \vec{B})^2 \right\rangle = \frac{1}{2}\vec{B}_{\text{Reg}}^2 + \frac{1}{3}\left\langle \delta\vec{B}^2 \right\rangle$$

with

$$W_B(k_{\text{Crit}}) = \int_{k_{\text{Crit}}}^{\infty} k P_B(k)\,\mathrm{d}k \quad \text{and} \quad W_N(k_{\text{Crit}}) = \int_{k_{\text{Crit}}}^{\infty} k P_N(k)\,\mathrm{d}k \quad (3.35)$$

Therefore we find that

$$\bar{\mathcal{Q}}_q = \mathcal{Q}_0 \cos(2\theta_{\text{Reg}}) \quad \text{and} \quad \bar{\mathcal{Q}}_u = \mathcal{Q}_0 \sin(2\theta_{\text{Reg}}) \quad (3.36)$$

while

$$\bar{\mathcal{Q}}_v = 0 \quad (3.37)$$

where

$$\mathcal{Q}_0 = \frac{1}{4}\left[\frac{2\vec{B}_{\text{Reg}}\omega}{M_F \bar{m}_{\text{Eff}}^2}\right]^2 I_N^3 - \frac{1}{4}\left[\frac{2\vec{B}_{\text{Reg}}\omega}{M_F \bar{m}_{\text{Eff}}^2}\right]^2 \cos\left(2\bar{\Delta}\right) + \frac{\vec{B}_{\text{Reg}}^2 L}{16M_F^2 \bar{n}_e^2} I_N^2 W_N(k_{\text{Crit}})$$

Thus we get an extra term in the Stokes' parameter which is proportional to L. Consequently, since $\bar{\Delta} \gg 1$, for cosmic microwave background photons in galaxy clusters this is the dominant term.

We let

$$P_B(k) \propto k^{\alpha-2} \qquad \text{and} \qquad P_N(k) \propto k^{\alpha-2} \tag{3.38}$$

when $k \simeq k_{\rm Crit}$, for some $\alpha < -1$. We define C_K and C_N as per (19 and 20 in [32]) by

$$k^2 P_B(k) = 2C_K \left(\frac{k}{k_0}\right)^{\alpha} \tag{3.39}$$

and

$$k^2 P_N(k) = 2(2\pi)^{\frac{1}{3}} C_N^2 k^{\alpha} \tag{3.40}$$

respectively, with $k_0 = 1\,\text{kpc}^{-1}$. Then performing the integrals (3.35) yields

$$W_B(k_{\rm Crit}) \simeq 2\frac{C_K}{|\alpha|}\left(\frac{k_{\rm Crit}}{k_0}\right)^{\alpha} \qquad \text{and} \qquad W_N(k_{\rm Crit}) \simeq 2\frac{(2\pi)^{\frac{1}{3}}}{|\alpha|} C_N^2 k_{\rm Crit}^{\alpha} \tag{3.41}$$

respectively. Therefore we get

$$k_{\rm Crit}^{-1} \simeq 2.4\times10^{-2}\text{pc}\left(\frac{\nu}{100\,\text{GHz}}\right)\left(\frac{10^{-3}\text{cm}^{-3}}{\bar{n}_e}\right) \tag{3.42}$$

where

$$\nu = \frac{\omega}{2\pi}$$

However, for galaxy clusters we have approximation (3.22) for the electron number density, while for cosmic microwave background photons the frequency satisfies $30\,\text{GHz} < \nu < 300\,\text{GHz}$. Hence we find that the critical scale satisfies $10^{-3}\text{pc} \lesssim k_{\rm Crit} \lesssim 10^{-1}\text{pc}$.

3.4 The Magnetic Fields of Galaxy Clusters

The space between the individual galaxies in a galaxy cluster is filled with a hot plasma. This gives rise to a magnetic field. These intra-cluster magnetic fields may be up to $30\,\mu$G, where $1\,\text{G} = 1\times10^{-4}$T.

There are two main types of galaxy cluster. The first is the *Non-Cooling Core* galaxy cluster. These have a magnetic field of about $5\pm5\,\mu$G which is magnetically ordered on a scale of about 15 ± 5 kpc. The second type are the *Cooling Core* galaxy clusters. These tend to have a larger magnetic field of about $20\pm10\,\mu$G which, however, is magnetically ordered on a smaller scale of about 5 ± 5 kpc.

The magnetic field of galaxy clusters is determined by the observation of the *Faraday Rotation Measure*, $F_{\rm RM}$, given by

$$F_{\rm RM}(z_s\hat{z}) = a_0 \int_0^{z_s} n_e(x\hat{z}) B_{\|}(x\hat{z})\, \mathrm{d}x \tag{3.43}$$

where

$$a_0 = \frac{\alpha_{\rm EM}^3}{\sqrt{\pi}\, me^2}$$

z_s is the location of the source, and $B_{\|}$ is the magnetic field along the line of sight. For a uniform magnetic field we expect that

$$\langle F_{\rm RM}\rangle = a_0 B_{\|} n_e L$$

However the dispersion of the Faraday rotation measures from extended radio sources implies that it is not in fact a realistic model for the intra-cluster magnetic field[32]. Rather, the intra-cluster magnetic fields have a component which fluctuates on small scales. Thus we use the cell model. Therefore, the rotation measure is a Gaussian distributed random variable with zero mean and variation

$$\langle F_{\rm RM}^2\rangle - \langle F_{\rm RM}\rangle^2 \propto \frac{1}{2} a_0^2 L_{\rm Coh} \int_0^L \left\langle \vec{B}^2(z)\right\rangle n_e^2(z)\, \mathrm{d}z \tag{3.44}$$

On the other hand, the power spectrum model is more realistic. We find that (18 in [32])

$$\langle F_{\rm RM}^2\rangle - \langle F_{\rm RM}\rangle^2 \propto \frac{1}{2} a_0^2 L_B \int_0^L \left\langle \vec{B}^2(z)\right\rangle n_e^2(z)\, \mathrm{d}z \tag{3.45}$$

Hence we must have

$$L_B = L_{\rm Coh}$$

for these to match.

For galaxy clusters free from strong radio halos and widespread cooling flows the average magnetic field strength and coherence length satisfy

$$\langle |B|\rangle_{\rm ICM} = 7.5 \pm 2.5 \left[\frac{L_{\rm dom}}{10\,\rm kpc}\right]^{-\frac{1}{2}} h_{75}^{\frac{1}{2}}\, \mu\mathrm{G} \tag{3.46}$$

where $H_0 = 75 h_{75}\,\mathrm{km\ s^{-1}Mpc^{-1}}$. Here cooling-core galaxy clusters are excluded.

X-ray surface brightness observations give a determination of the electron number density. By ROSAT (26 in [32])

$$n_e(r) = n_0 \left[1 + \frac{r^2}{r_c^2}\right]^{-\frac{3\beta}{2}} \tag{3.47}$$

where β here is of order one. Furthermore, by *Magneto-Hydrodynamic Simulations* of galaxy clusters[32] we find that

$$\left\langle \vec{B}^2 \right\rangle^{\frac{1}{2}} \propto \langle n_e \rangle^\eta$$

with

$$\eta = 1 \tag{3.48}$$

However, from observations of the galaxy cluster *Abell 119* (27 in [32])

$$\eta = 0.9 \tag{3.49}$$

On the other hand, the theoretical prediction if the magnetic field is frozen in[32] is

$$\eta = \frac{2}{3} \tag{3.50}$$

An alternative theoretical prediction is given if the total magnetic energy scales as the thermal energy as per (26 in [32]), whence

$$\eta = \frac{1}{2} \tag{3.51}$$

By the observation of Faraday rotation measures (25 in [32]) the magnetic field of galaxy clusters extends as far as the ROSAT detectable X-ray emission coming purely from the electron number density. Therefore the scale is about 500 kpc[32].

We recall that by Kolmogorov[39] the power spectrum of three-dimensional turbulence on small scales is universal and is given by

$$P(k) \propto k^{-\frac{11}{3}} \tag{3.52}$$

By (20 in [32])

$$\alpha \simeq -\frac{5}{3} \tag{3.53}$$

for spatial scales satisfying $10^6\,\mathrm{m} < k^{-1} < 10^{13}\,\mathrm{m}$ with $C_N^2 \simeq 10^{-3}\mathrm{m}^{-\frac{20}{3}}$.

Furthermore, by (28 in [32]) we also get the condition (3.53) for spatial scales satisfying $10^{-2}\mathrm{pc} \lesssim k^{-1} \lesssim 3.6\,\mathrm{pc}$ and $C_K \simeq 2.93 \times 10^6\,\mathrm{kg\ s}^{-2}$.

On larger scales according to (19 in [32]) the magnetic flux power spectrum flattens with

$$\alpha \simeq -0.37 \pm 0.1 \tag{3.54}$$

for spatial scales satisfying $0.5\,\mathrm{kpc} \lesssim k^{-1} \lesssim 15\,\mathrm{kpc}$ and $C_K \simeq 2.1 \times 10^6\,\mathrm{kg\ s}^{-2}$.

By (17 in and 18 in [32]) we get information about the power spectrum of the galaxy cluster magnetic field fluctuations from the Faraday rotation measure images which imply that

$$-2 < \alpha < 0 \tag{3.55}$$

In addition, by (29 in [32]), for $k \gtrsim 1\,\mathrm{kpc}^{-1}$ we find that

$$-2 < \alpha < -1.6 \tag{3.56}$$

Therefore this is consistent with Kolmogorov turbulence on small scales. However, if

$$k \lesssim 1\,\mathrm{kpc}^{-1}$$

then we find that $k^2 P_B(k)$ is flat or increasing with k.

The mixing between the chameleon field and the photon field is sensitive[32] to $P_B(k)$ and $P_N(k)$ on scales of k_{Crit}^{-1}, where for the cosmic microwave background in galaxy clusters

$$10^{-3}\,\mathrm{pc} \lesssim k_{\mathrm{Crit}}^{-1} < 0.1\,\mathrm{pc}$$

We let α be given by that of Kolmogorov turbulence below k_{Crit}^{-1}. The dominant contribution to $\left\langle \delta\vec{B}^2 \right\rangle$ comes from the scale where

$$\eta = 3 \tag{3.57}$$

with

$$P_B(k) \propto k^{-\eta} \tag{3.58}$$

as k increases. This gives a scale k_*, say, such that

$$k^2 P_B(k) = 2C_K \left(\frac{k}{k_0}\right)^{-\frac{5}{3}} \tag{3.59}$$

for $k > k_*$ such that we get Kolmogorov turbulence. Meanwhile we let

$$k^2 P_B(k) = 2C_K \left(\frac{k_*}{k_0}\right)^{-\frac{5}{3}} \left(\frac{k}{k_*}\right)^{\beta} \tag{3.60}$$

for $k_{\min} < k < k_*$, where $\beta > -1$ and we ignore the contribution of $k^2 P_B(k)$ when $k < k_{\min}$. That is, we let

$$\lim_{k \to 0} k^2 P_B(k) \to C \qquad \text{and} \qquad \lim_{k \to 0} k^2 P_N(k) \to \tilde{C} \tag{3.61}$$

where C and $\tilde{C}$ are some constants. Thus

$$\begin{aligned} \left\langle \delta \vec{B}^2 \right\rangle &= \frac{1}{4\pi} \int \mathrm{d}^3 k \, P_B(k) \\ &= \left[2 f_{1+\beta}(x) + 3\right] C_K k_* \left(\frac{k_*}{k_0}\right)^{-\frac{5}{3}} \end{aligned} \tag{3.62}$$

and

$$\begin{aligned} L_B \left\langle \delta \vec{B}^2 \right\rangle &= \frac{1}{2} \int k P_B(k) \, \mathrm{d}k \\ &= \left[f_\beta(x) + \frac{3}{5}\right] C_K \left(\frac{k_*}{k_0}\right)^{-\frac{5}{3}} \end{aligned} \tag{3.63}$$

where

$$x = \frac{k_{\min}}{k_*}$$

with

$$f_\beta = \frac{1}{\beta}\left[1 - x^\beta\right]$$

Hence substituting (3.62) into (3.63) gives

$$k_* = L_B^{-1} \left[\frac{\left(f_\beta(x) + \frac{3}{5}\right)}{\left(2 f_{1+\beta}(x) + 3\right)}\right] \tag{3.64}$$

Furthermore, substituting (3.64) back into (3.63) we find that

$$k_0^{\frac{5}{3}} C_K = g(x) \left\langle \delta \vec{B}^2 \right\rangle L_B^{-\frac{2}{3}}$$

where

$$g(x) = \frac{\left[f_\beta(x) + \frac{3}{5}\right]^{\frac{2}{3}}}{\left[2 f_{1+\beta}(x) + 3\right]^{\frac{5}{3}}}$$

On large scales of $100\,\mathrm{kpc}^{-1} \lesssim k \lesssim 10\,\mathrm{kpc}^{-1}$ by (29 in [32]) we get

$$\beta = 0 \tag{3.65}$$

We let $k_{\min} \simeq L_{\mathrm{Clust}}^{-1}$, where L_{Clust} is the scale of the magnetic region. Moreover, we let $k_* \simeq L_B^{-1}$ whence

$$10 \lesssim \frac{k_*}{k_{\min}} \lesssim 200$$

In this small β limit we see that

$$\lim_{\beta \to 0} f_\beta(x) = -\log(x)$$

where we have written x^β as $\exp[\beta \log(x)]$ and used L'Hôpital's Rule, along with

$$\lim_{\beta \to 0} f_{1+\beta}(x) = 1 - x$$

Therefore with $\beta = 0$ as given by (3.65) we find that[32]

$$\frac{5^{\frac{7}{3}}}{4} g(x) = \left[\frac{5}{8} \log\left(\frac{k_*}{k_{\min}}\right) + \frac{3}{8}\right]^{\frac{2}{3}} \tag{3.66}$$

or

$$0.14 \lesssim g(x) \lesssim 0.22 \tag{3.67}$$

Hence

$$0.14 \lesssim \frac{k_0^{\frac{5}{3}} C_K}{\left\langle \delta \vec{B}^2 \right\rangle L_B^{-\frac{2}{3}}} \lesssim 0.22 \tag{3.68}$$

and so

$$0.7 \lesssim \frac{C_K}{3 \times 10^6\,\mathrm{kg\ s}^{-2}} \lesssim 2.2 \tag{3.69}$$

for a galactic magnetic field with $\left\langle \delta \vec{B}^2 \right\rangle \simeq 3\,\mu\mathrm{G}$ and $20\,\mathrm{pc} \lesssim L_B \lesssim 50\,\mathrm{pc}$.

3.5 The Thermal Sunyaev-Zel'Dovich Effect

By (30 in [32]) the fractional change in the cosmic microwave background temperature due to the thermal Sunyaev-Zel'Dovich effect for a photon passing through a galaxy cluster is

$$\frac{\Delta T_{\mathrm{SZ}}}{T_0} = \frac{k_B T_e}{m_e} \tau_0 \left[x \coth\left(\frac{x}{2}\right) - 4\right] \tag{3.70}$$

where

$$\tau_0 = \int \sigma_T n_e(l)\,\mathrm{d}l$$

is the optical depth, and $\sigma_T \simeq 6.65 \times 10^{-29}\mathrm{m}^2$ is the Thompson scattering cross-section for an electron in a plasma of temperature T_e. At low frequencies this causes a decrease in the temperature of the cosmic microwave background. On the other hand, at high frequencies we find an increase in the temperature of the cosmic microwave background.

We ignore higher order corrections to the Sunyaev-Zel'Dovich effect due to relativistic high velocity effects, for example.

3.6 The Modified Chameleonic Sunyaev-Zel'Dovich Effect

Since the intrinsic polarisation of the cosmic microwave background (3.19) and (3.20) are so small, while

$$\left\langle \mathcal{P}^2_{\gamma\leftrightarrow\phi} \right\rangle^{\frac{1}{2}} \simeq \bar{\mathcal{P}}_{\gamma\leftrightarrow\phi} \tag{3.71}$$

we may put

$$\left\langle \mathcal{Q}^2_i \right\rangle^{\frac{1}{2}} \simeq \bar{\mathcal{P}}_{\gamma\leftrightarrow\phi} \tag{3.72}$$

for each i. Then for the intensity (3.14) we get

$$I_\gamma(z) \simeq I(0)\left[1 - \bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}(L)\right] \tag{3.73}$$

This yields a change in the cosmic microwave background intensity due to the chameleonic Sunyaev-Zel'Dovich effect of

$$\frac{\Delta I_{\mathrm{CSZ}}}{I_0} = -\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}(L) \tag{3.74}$$

Therefore, in terms of temperature we get a change

$$\begin{aligned} \frac{\Delta T_{\mathrm{CSZ}}}{T_0} &= \frac{1-\exp(-x)}{x}\frac{\Delta I_{\mathrm{CSZ}}}{I_0} \\ &= \frac{1-\exp(-x)}{x}\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}(L) \end{aligned} \tag{3.75}$$

By equation (3.24) we see that

$$\mathcal{P}_{\gamma\leftrightarrow\phi}(L) = \mathcal{P}_{\gamma\leftrightarrow\phi}(L, n_e, \vec{B}, \omega) \tag{3.76}$$

We let $\left\langle \vec{B}^2 \right\rangle^{\frac{1}{2}} \propto \langle n_e \rangle^{\eta}$, for some η. Then for the cell model we get

$$\frac{\Delta I_{\text{CSZ}}}{I_0} \propto n_e^{-2(1-\eta)} \omega^2 \tag{3.77}$$

Furthermore, for the power spectrum model, with $P_B(k)$ and $P_N(k)$ given by (3.38), for $k \geq k_{\text{Crit}}$, by equation (3.42) we get

$$\frac{\Delta I_{\text{CSZ}}}{I_0} \propto n_e^{2\eta+\alpha} \omega^{-\alpha} \tag{3.78}$$

where $-2 < \alpha < -1$. If $\alpha < -2$ we get the same behaviour as that of the power spectrum model for high frequencies. However, for low frequencies we get the same behaviour as that of the cell model. For a Kolmogorov spectrum for the magnetic fluctuations we let α be given by equation (3.53). If $\eta = 0.9$, as given by equation (3.49), then the chameleonic Sunyaev-Zel'Dovich effect becomes proportional to $n_e^{\frac{2}{15}} \omega^{\frac{5}{3}}$. Contrastingly the thermal Sunyaev-Zel'Dovich effect is proportional to n_e^1. Furthermore, if $\alpha = -\frac{5}{3}$ and $\eta = 0.9$, as given by equations (3.53) and (3.49) respectively, then the change in the intensity $\frac{\Delta I_{\text{CSZ}}}{I_0}$ due to the chameleonic Sunyaev-Zel'Dovich effect has a shallower dependence on the electron number density than the change in the intensity $\frac{\Delta I_{\text{SZ}}}{I_0}$ due to the thermal Sunyaev-Zel'Dovich effect. Therefore we can distinguish the chameleonic Sunyaev-Zel'Dovich effect from the thermal Sunyaev-Zel'Dovich effect by looking at either the electron number density dependence or the frequency dependence. However by equations (3.77) and (3.78) we see that we must have a magnetic field correlation length and electron number density correlation length that do not vary with the electron number density. To this end we let

$$L_{\text{Coh}} = L_B \propto L_N \tag{3.79}$$

giving

$$\frac{\Delta I_{\text{CSZ}}}{I_0} \propto n_e^{2\eta+\alpha_0} \omega^{-\alpha_0} L_{\text{Coh}}^{1+\alpha_0} \tag{3.80}$$

where

$$\alpha = \alpha_0$$

for $-2<\alpha<-1$, while in the cell model

$$\alpha = -2 \tag{3.81}$$

However, if $\alpha < -2$ then this implies that the scaling with L_{Coh} is more complicated than that discussed so far. Hence we let $-2 \leq \alpha \leq -1$ for the power spectrum model.

3.7 The Magnitude of The Chameleonic Sunyaev-Zel'Dovich Effect

In the power spectrum model, by equation (3.42) for $k \geq k_{\mathrm{Crit}}$ we have $\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi} \propto \omega^{-\alpha}$ where $-2<\alpha<-1$. We let

$$\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}(L,\omega) = \bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}(L,\omega_0)\left(\frac{\omega}{\omega_0}\right)^{-\alpha} \tag{3.82}$$

for some fixed ω_0. We choose ω_0 such that if $-2 < \alpha < -1$ then $\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}(L,\omega_0)$ is independent of α.

By equations (3.24) and (3.34) a bound on $\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}(L,\omega_0)$ gives a constraint on the coupling g_F, where

$$g_F = \frac{1}{M_F}$$

We have the dependencies of $\bar{\mathcal{P}}_{\gamma\leftrightarrow\phi}$ as given by equation (3.76), where values for $\left\langle \vec{B}^2 \right\rangle$, $\langle n_e \rangle$, $\langle n_e^2 \rangle$, L_B, L_N, C_B and C_N are known from observation. This means that the accuracy to which these variables are know will affect the accuracy to which any constraint on g_F may be found. In addition, we note that g_F is a function of α.

We consider both the cell model with $\alpha = -2$ as in (3.81) and the Kolmogorov turbulence model with $\alpha = -\frac{5}{3}$ as in (3.53).

For the case of the cell model, from equation (3.24) we get

$$\bar{\mathcal{P}}^{(\mathrm{Cell})}_{\gamma\leftrightarrow\phi}(L,\nu) = \left(3.2\times 10^{-10}\right) g^2_{10^{-10}\mathrm{GeV}^{-1}} B^2_{10\mu\mathrm{G}} n^{-2}_{0.01} L_{200\mathrm{kpc}} \left(L^{(\mathrm{Coh})}_{1\mathrm{kpc}}\right)^{-1} \nu^2_{214\mathrm{GHz}}$$

where

$$\begin{aligned} B_{10\mu\mathrm{G}} &= \frac{1}{10\,\mu\mathrm{G}}\left\langle \vec{B}^2\right\rangle^{\frac{1}{2}} \\ n_{0.01} &= \frac{1}{10^{-2}\,\mathrm{cm}^{-3}}n_e \\ \nu_{214\mathrm{GHz}} &= \frac{1}{214\,\mathrm{GHz}}\nu \\ L_{200\mathrm{kpc}} &= \frac{1}{200\,\mathrm{kpc}}L \\ L_{1\mathrm{kpc}}^{(\mathrm{Coh})} &= \frac{1}{1\,\mathrm{kpc}}L_{\mathrm{Coh}} \end{aligned}$$

and

$$g_{10^{-10}\mathrm{GeV}^{-1}} = \frac{1}{10^{-10}\,\mathrm{GeV}^{-1}}g_F$$

when (3.28) holds such that

$$\sin^2\left(\frac{\bar{\Delta}}{N}\right) \simeq \frac{1}{2}$$

On the other hand, for the power spectrum model with the magnetic field and the electron number density power spectra following a Kolmogorov law for $L_{\mathrm{Coh}}^{-1} < k < k_{\mathrm{Crit}}$ by equation (3.42), where for $k \leq L_{\mathrm{Coh}}^{-1}$ with $\eta \simeq 0$, and $\eta > 0$, we see that the dominant contribution to the magnetic field fluctuation comes from spatial scales of about L_{Coh}.

Then with the bounds (3.68) by equation (3.34) we get

$$2.4\times10^{-7} \lesssim \frac{\bar{\mathcal{P}}^{(\mathrm{Kolmo})}_{\gamma\leftrightarrow\phi}(L,\nu)}{g^2_{10^{-10}\mathrm{GeV}^{-1}}B^2_{10\mu\mathrm{G}}n_{0.01}^{-\frac{5}{3}}\nu_{214\mathrm{GHz}}^{\frac{5}{3}}L_{200\mathrm{kpc}}\left(L_{1\mathrm{kpc}}^{(\mathrm{Coh})}\right)^{-\frac{2}{3}}\left[\frac{I_N^2(2I_N-1)}{12}\right]} \lesssim 3.8\times10^{-7} \tag{3.83}$$

where $1 \lesssim I_N \lesssim 2$ such that $1 \lesssim I_N^2(2I_N - 1) \lesssim 12$.

By equation (3.46), with $1\,\mathrm{kpc} \lesssim L_{\mathrm{Coh}} \lesssim 100\,\mathrm{kpc}$ for a non-cooling core galaxy cluster in the cell model we find that

$$8.0\times10^{-14} \lesssim \frac{\bar{\mathcal{P}}^{(\mathrm{Cell})}_{\gamma\leftrightarrow\phi}(L,\nu)}{g^2_{10^{-10}\mathrm{GeV}^{-1}}n_{0.01}^{-\frac{5}{3}}\nu_{214\mathrm{GHz}}^{\frac{5}{3}}L_{200\mathrm{kpc}}} \lesssim 3.2\times10^{-9} \tag{3.84}$$

while for the Kolmogorov power spectrum model we get

$$2.2\times10^{-11} \lesssim \frac{\bar{\mathcal{P}}^{(\mathrm{Kolmo})}_{\gamma\leftrightarrow\phi}(L,\nu)}{g^2_{10^{-10}\mathrm{GeV}^{-1}}n_{0.01}^{-\frac{5}{3}}\nu_{214\mathrm{GHz}}^{\frac{5}{3}}L_{200\mathrm{kpc}}} \lesssim 3.8\times10^{-6} \tag{3.85}$$

On the other hand cooling-core galaxy clusters tend to have higher magnetic field strength. Therefore there will be a greater chameleonic Sunyaev-Zel'Dovich effect.

By (21 in [32]) the galaxy cluster *Hydra A* has strong cooling flows. The Faraday rotation measure from an embedded radio source gives a smooth magnetic field component $\vec{B}_{\mathrm{Reg}}$ of $6\,\mu$G on a scale of 100 kpc and a fluctuating magnetic field component $\delta\vec{B}$ of $30\,\mu$G which changes on a scale of 4 kpc.

By X-ray data[32] we find that the galaxy core radius, R_{Core}, is about $R_{\mathrm{Core}} = 130\,\mathrm{kpc}$ while the average electron number density is about $\langle n_e\rangle = 10^{-2}\mathrm{cm}^{-3}$.

We let

$$\begin{aligned} L &= L_{\mathrm{Core}} \\ &= 2R_{\mathrm{Core}} \\ &= 260\,\mathrm{kpc} \end{aligned}$$

Then by the bounds (3.84) and (3.85) for the galaxy cluster *Hydra A* we get

$$\bar{\mathcal{P}}^{(\mathrm{Cell})}_{Hydra\ A} = (0.93\times 10^{-9})g^2_{10^{-10}\mathrm{GeV}^{-1}}\nu^2_{214\mathrm{GHz}} \tag{3.86}$$

and

$$0.97\times 10^{-7} \leq \frac{\bar{\mathcal{P}}^{(\mathrm{Kolmo})}_{Hydra\ A}}{g^2_{10^{-10}\mathrm{GeV}^{-1}}\nu^{\frac{5}{3}}_{214\mathrm{GHz}}} \leq 17.9\times 10^{-7} \tag{3.87}$$

respectively. On the other hand, for a non-cooling core galaxy cluster we find that there is little change to the chameleon-photon interaction probability.

By (32 in [32]) from the Faraday rotation measure of a radio source at the centre of the *Coma* cluster we find a smooth magnetic field component $\vec{B}_{\mathrm{Reg}}$ of $0.2 \pm 0.1\,\mu$G on a scale of 200 kpc and a fluctuating magnetic field component $\delta\vec{B}$ of $8.5 \pm 1.5\,\mu$G which changes on a scale of 1 kpc. Thus with

$$\begin{aligned} L &= L_{\mathrm{Core}} \\ &= 198\,\mathrm{kpc} \end{aligned}$$

and $\langle n_e\rangle = 4\times 10^{-3}\mathrm{cm}^{-3}$ we find that

$$\bar{\mathcal{P}}^{(\mathrm{Cell})}_{Coma} = (1.4\times 10^{-9})g^2_{10^{-10}\mathrm{GeV}^{-1}}\nu^2_{214\mathrm{GHz}} \tag{3.88}$$

while

$$0.64\times 10^{-7} \leq \frac{\bar{\mathcal{P}}^{(\text{Kolmo})}_{Coma}}{g^2_{10^{-10}\text{GeV}^{-1}}\nu^{\frac{5}{3}}_{214\text{GHz}}} \leq 12.79\times 10^{-7} \tag{3.89}$$

Therefore the magnitude of the chameleonic Sunyaev-Zel'Dovich is similar for the *Hydra A* galaxy cluster and the *Coma* galaxy cluster.

By (33 in [32]) we have a range of Sunyaev-Zel'Dovich effect measurements for the *Coma* galaxy cluster.

By equation (3.70) the Sunyaev-Zel'Dovich effect depends on the optical depth τ_0 and the electron temperature in the plasma core. For the *Coma* galaxy cluster

$$k_B T_e \simeq 8.2\,\text{keV}$$

Including the chameleonic Sunyaev-Zel'Dovich effect, where $\Delta T_{\text{CSZ}} \propto \omega^{-\alpha}$, we get

$$\frac{\Delta T}{T_0} = 1.6\times 10^{-2}\tau_0 \left[x\coth\left(\frac{x}{2}\right) - 4\right] + \left[\exp(-x) - 1\right] x^{-\alpha_0 - 1}\left(\frac{k_B T_e}{m_e}\right)^{-\alpha_0} \bar{\mathcal{P}}^{(Coma)}_{\gamma\leftrightarrow\phi}(\omega_0) \tag{3.90}$$

where

$$x = \frac{\omega}{k_B T_0}$$

and $\alpha_0 = -\frac{5}{3}$ as given by (3.53) if $-2<\alpha<-1$ for the power spectrum model while $\alpha_0 = -2$ as in (3.81) for the cell model.

We maximise the likelihood L over the optical depth τ_0 and $\bar{\mathcal{P}}^{(Coma)}_{\gamma\leftrightarrow\phi}(\omega_0)$, where

$$-2\log(L) = \sum_i \frac{\left[\Delta T_i^{\text{observ}} - \Delta T_i\left(\tau_0, \bar{\mathcal{P}}^{(Coma)}_{\gamma\leftrightarrow\phi}\right)\right]^2}{\sigma_i^2}$$

where the $\Delta T_i^{\text{observ}}$ are the observed values of the temperature at frequencies of

$$\nu = \frac{\omega_i}{2\pi}$$

with standard error σ_i. For the best fit values $\hat{\tau}_0$ and $\hat{\bar{\mathcal{P}}}^{(Coma)}_{\gamma\leftrightarrow\phi}$ we let

$$\chi^2\left(\tau_0, \bar{\mathcal{P}}^{(Coma)}_{\gamma\leftrightarrow\phi}\right) = -2\log\left[\frac{L\left(\tau_0, \bar{\mathcal{P}}^{(Coma)}_{\gamma\leftrightarrow\phi}\right)}{L\left(\hat{\tau}_0, \hat{\bar{\mathcal{P}}}^{(Coma)}_{\gamma\leftrightarrow\phi}\right)}\right] \tag{3.91}$$

be given by a χ^2-distribution.

For $\omega_0 = 0.844\,\text{meV}$, that is $\nu = 204\,\text{GHz}$, the constraint on $\bar{\mathcal{P}}^{(Coma)}_{\gamma\leftrightarrow\phi}$ is independent of the parameter α_0.

Maximising $\chi^2\left(\tau_0, \bar{\mathcal{P}}^{\,(Coma)}_{\gamma\leftrightarrow\phi}\right)$ with respect to $\bar{\mathcal{P}}^{\,(Coma)}_{\gamma\leftrightarrow\phi}$ gives a value for τ_0 of

$$\tau_0 = (4.7\pm1.0)\times10^{-3} \tag{3.92}$$

Therefore to 95% confidence we find that

$$\bar{\mathcal{P}}^{\,(Coma)}_{\gamma\leftrightarrow\phi} < 6.2\times10^{-5} \tag{3.93}$$

By equation (3.88) we find that

$$\bar{\mathcal{P}}^{(\mathrm{Cell})}_{Coma}(204\,\mathrm{GHz}) = (1.3\times10^{-9})g^2_{10^{-10}\mathrm{GeV}^{-1}}\left(\frac{B_{\mathrm{Rand}}}{8.5\,\mu\mathrm{G}}\right)^2\left(\frac{4\times10^{-3}\mathrm{cm}^{-3}}{n_e}\right)^{-\frac{5}{3}}\left(\frac{L}{198L_{\mathrm{Coh}}}\right)$$

Moreover, by the bounds (3.89) we find that

$$0.59\times10^{-7} \leq \frac{\bar{\mathcal{P}}^{(\mathrm{Kolmo})}_{Coma}(204\,\mathrm{GHz})}{g^2_{10^{-10}\mathrm{GeV}^{-1}}\left(\frac{B_{\mathrm{Rand}}}{8.5\,\mu\mathrm{G}}\right)^2\left(\frac{4\times10^{-3}\mathrm{cm}^{-3}}{n_e}\right)^{-\frac{5}{3}}\left(\frac{L}{198\,\mathrm{kpc}}\right)\left(\frac{L_{\mathrm{Coh}}}{1\,\mathrm{kpc}}\right)^{-\frac{2}{3}}} \leq 11.8\times10^{-7}$$

By observation

$$B_{\mathrm{Rand}} = 8.5\,\mu\mathrm{G}$$
$$L = 198\,\mathrm{kpc}$$
$$L_{\mathrm{Coh}} = 1\,\mathrm{kpc}$$

and

$$n_e = 4\times10^{-3}\mathrm{cm}^{-3}$$

Hence to 95% confidence we get

$$g_{\mathrm{F}}^{(\mathrm{Cell})} < 2.2\times10^{-8}\,\mathrm{GeV}^{-1} \tag{3.94}$$

for a magnetic field described by the cell model, or

$$g_{\mathrm{F}}^{(\mathrm{Kolmo})} < 7.2\times10^{-10}\,\mathrm{GeV}^{-1} \tag{3.95}$$

for a turbulent Kolmogorov power spectrum model magnetic field. The stronger of these bounds, (3.95), corresponds to a bound on M_F of

$$M_F > 1.1 \times 10^9 \, \text{GeV} \tag{3.96}$$

This agrees with the bound obtained by considering the polarisation of starlight due to the chameleon field.

Chapter 4

Conclusions

4.1 The Chameleon Model

To conclude a brief recap of some of the important results highlighted in this exposition are given.

Firstly, recall that current astro-physical observations suggest that our universe is currently undergoing an era of accelerated cosmic expansion. This motivates the introduction of dark energy, modelled as a negative pressure cosmological constant in the standard cosmology.

An alternative approach is to consider a dynamic scalar field, usually called quintessence, which gives a small vacuum energy density in the late universe. This would then give rise to accelerating cosmic expansion. Such a field would couple to all matter fields gravitationally. However, such a field has not yet been detected, with a slim field being ruled out in many cases.

It it worthwhile to repeat the consequences of taking the novel approach of considering a scalar field which has a potential energy function that is a function of the surrounding matter density. This gives rise to a mass term for the scalar field which is itself a function of the surrounding matter density. Hence the mass of the field changes with its environment, motivating the name chameleon scalar field. This scalar field may couple to the Standard Model matter fields gravitationally yet avoid experimental detection.

Due to the bounds set by experimental searches for fifth-force interactions, the scale of the chameleon theory must satisfy $M \lesssim 10^{-3}\,\mathrm{eV}$, as set by equation (2.17). Then performing an expansion of the effective potential energy function of the theory as in equation (2.13) it is found that a constant term $M^4 \simeq 10^{-12}\,\mathrm{eV}$ of the same order as the vacuum energy density is obtained, as shown in equation (2.19).

4.2 The Cosmic Microwave Background

After much work, it was found that indeed the interaction of a chameleon field with the photon may give rise to a detectable effect. In the presence of the magnetic field of a galaxy cluster, interactions between the chameleon field and the photon field of the cosmic microwave background are predicted to give a chameleonic Sunyaev-Zel'Dovich effect. That is, these interactions would give rise to anisotropies in the cosmic microwave background.

The usual thermal Sunyaev-Zel'Dovich effect occurs from the interaction of cosmic microwave background photons with high energy electrons in the hot plasma core of a galaxy cluster. Thus warm spots are found in the cosmic microwave background in the direction of galaxy clusters.

With the addition of a coupling between the chameleon field and the photon field to the scalar-tensor action of the chameleon theory, an expression for the change to the cosmic microwave background was found in equation (3.75). The exact magnitude of the effect is dependent on the model used to describe the magnetic field of the galaxy cluster. For two such models, the cell model and power spectrum model respectively, bounds were found on the theoretical chameleon-photon mixing probability, namely equations (3.84) and (3.85). These bounds assume a non-cooling core galaxy cluster.

Using the *Coma* galaxy cluster as an example, these bounds on the probability of interactions between the chameleon field and the photon field can be used to place constraints on the chameleon field theory. The *Coma* galaxy cluster was chosen as there exist many measurements of its magnetic field and the anisotropies it causes on the cosmic microwave background via the Sunyaev-Zel'Dovich effect. To 95% confidence it was found that the coupling of the chameleon-photon interaction must satisfy $g_{\mathrm{F}}^{(\mathrm{Cell})} < 2.2{\times}10^{-8}\,\mathrm{GeV}^{-1}$ using the cell model for the magnetic field, or $g_{\mathrm{F}}^{(\mathrm{Kolmo})} < 7.2\times 10^{-10}\,\mathrm{GeV}^{-1}$ using the power spectrum model for the magnetic field, equations (3.94) and (3.95) respectively. The stronger of these bounds, (3.95), corresponds to a constraint on the scale of the chameleon theory of $M_F > 1.1\times 10^{9}\,\mathrm{GeV}$, by equation (3.96).

4.3 Future Work

The novel approach of a chameleon scalar field has been analysed for both the case of a cyclic universe and a universe with large extra dimensions[40, 41].

A *Cyclic Model* of the universe[42, 43, 44] gives an alternative to cosmic inflation, yet still solves the Cosmic Homogeneity and Isotropy Problem, the Cosmic Flatness Problem, and the Magnetic Monopole Problem, while generating a spectrum of density fluctuations which may give rise to Large-Scale-Structure Formation. In such a theory it is postulated that the Big Bang is not the beginning of time. Rather, the evolution of the universe is cyclic. The universe is claimed to consist of two branes, which are separated by a microscopic gap. The observable Standard Model particles live on one of these branes. The other brane consists of known or unknown particles which may only interact with the particles of the first brane gravitationally. A scalar field ϕ, say, measures the inter-brane separation in a higher dimension. Then the Big Bang is proposed to be collisions between these two branes. After a collision, a period of rapidly expanding inter-brane separation occurs. The branes start to slow, and the universe can assume both a radiation dominated era and a matter dominated era. In addition large-scale-structure may then be formed. Eventually the branes come to a stop. At this stage, the potential energy between the branes causes accelerated cosmic expansion. During this stage of accelerated cosmic expansion the branes are stretched out, and so the matter, radiation and large-scale-structure become diluted. Hence the branes approach a vacuum state. The acceleration stops, and a contraction in the inter-brane separation begins. This leads to another brane collision, and the process repeats. Quantum fluctuations during the period of contracting inter-brane separation give rise to anisotropies in the following cycle of the universe which may grow into the large-scale-structure.

In the latter theory, known as the *Randall Sundrum Model*[45], a fifth dimension is postulated which is non-compact. Such an undetected extra dimension is possible if the 5-dimensional metric is non-factorisable. In this case, there may be a bound state of the graviton in the higher dimension. It is proposed that our universe lives on a 3-brane in a 5-dimensional universe, with the fifth dimension having a boundary at some finite distance. Then the coupling of the fields are governed by these boundary conditions. A Kaluza-Klein reduction[46] of the 5-dimensional metric can give rise to one scalar component which is a quantum mechanical wave-function in the fifth dimension, and a 4-dimensional graviton.

The ϕ field which describes the inter-brane separation of the cyclic universe may be modelled as a chameleon field. In this case, the strict bound on the coupling of the ϕ field to the Standard Model matter fields is greatly relaxed, since, as discussed at length already, a chameleon field may avoid experimental detection.

In addition, the radion field ϕ of the Randall-Sundrum model may be a chameleon field. Once again, the gravitational coupling of the radion to the standard model matter fields is predicted to be too large, as it would violate experimental bounds set by searches for a fifth force. However the chameleon field may yet again prove fruitful in saving such a theory from experimental contradictions by giving a radion field which would not be expected to have yet been detected by current experiments.

These, and other possibilities, await a description with the addition of a chameleon scalar field.

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