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Non-commutative automorphisms of bounded non-commutative domains

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We establish rigidity (or uniqueness) theorems for non-commutative (NC) automorphisms that are natural extensions of classical results of H. Cartan and are improvements of recent results. We apply our results to NC domains consisting of unit balls of rectangular matrices.

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1. Introduction

Holomorphic automorphisms of domains in $\mathbb{C}^d$ have been studied since the work of H. and E. Cartan in the 1930s [6, 8]. A holomorphic function can be thought of as a generalized polynomial, and they can be evaluated not just on tuples of complex numbers, but also on tuples of commuting matrices or commuting operators whose spectrum is in the domain of the function [21]. A non-commutative (NC) function is a generalization of a free polynomial (i.e. a polynomial in non-commuting variables), and it is natural to evaluate them on tuples of matrices or operators.

To describe NC functions (following [13], for example), we must first establish some notation. Let $\mathbb{M}_n$ denote the $n \times n$ complex matrices, and let $\mathbb{M}_n^d$ denote the d-tuples of $n \times n$ matrices. We shall let $\mathbb{M}^{[d]}$ denote the disjoint union $\bigcup_{n=1}^{\infty} \mathbb{M}_n^d$. Given $x = (x^1, \ldots, x^d)$ in $\mathbb{M}_n^d$ and $y = (y^1, \ldots, y^d)$ in $\mathbb{M}_m^d$, by $x \oplus y$ we mean the element $(x^1 \oplus y^1, \ldots, x^d \oplus y^d)$ of $\mathbb{M}_{m+n}^d$. If $x \in \mathbb{M}_n^d$ and $s, t \in \mathbb{M}_n$, by sxt we mean $(sx^1t, \ldots, sx^dt)$.

A free polynomial p in d variables can be thought of as a function defined on $\mathbb{M}^{[d]}$, and as such it has the following properties:

(i) if x is in $\mathbb{M}_n^d$, then $p(x) \in \mathbb{M}_n$;

(ii) if x and y are in $\mathbb{M}^{[d]}$, then $p(x \oplus y) = p(x) \oplus p(y)$;

(iii) if $x \in \mathbb{M}_n^d$ and $s \in \mathbb{M}_n$ is invertible, then $p(s^{-1}xs) = s^{-1}p(x)s$.

Definition 1.1. An NC set is a set $\Omega \subseteq \mathbb{M}^{[d]}$ such that $\Omega_n := \Omega \cap \mathbb{M}_n^d$ is an open set for each n, and such that Ω is closed with respect to direct sums and joint unitary equivalence (i.e. for all $x \in \Omega_n$ and for all u unitary in $\mathbb{M}_n$, we have $u^{-1}xu \in \Omega$). If an NC set Ω has the property that Ω_n is connected for every n, we shall call it an NC domain.

An NC function is a function on an NC set that mimics the properties (i)–(iii) above of free polynomials.

Definition 1.2. An NC function f on an NC set Ω is a function with the following three properties:

(i) if x is in Ω_n, then $f(x) \in \mathbb{M}_n$ (we say that f is graded if this occurs);

(ii) if x and y are in Ω, then $f(x \oplus y) = f(x) \oplus f(y)$;

(iii) if $x \in \Omega_n$, $s \in \mathbb{M}_n$ is invertible and $s^{-1}xs \in \Omega$, then $f(s^{-1}xs) = s^{-1}f(x)s$.

An NC map Φ on an NC domain $\Omega \subseteq \mathbb{M}^{[d]}$ is a d-tuple of NC functions. If Φ is an NC map on Ω that is also a bijection onto Ω, we call it an NC automorphism.

Bounded symmetric domains in $\mathbb{C}^d$ have been characterized by E. Cartan [8], and in the course of the proof automorphisms of such domains were described.

We are interested in the following questions about NC automorphisms.

Question 1.3 (rigidity). If Φ is an NC automorphism of Ω, is it uniquely determined by its action on $\Omega \cap \mathbb{M}_n^d$ for some fixed n?

Question 1.4 (extendibility). If $F\colon \Omega \cap \mathbb{M}_n^d \to \Omega \cap \mathbb{M}_n^d$ is a biholomorphic map that respects similarities, is there an NC automorphism $\Phi\colon \Omega \to \Omega$ such that $\Phi|_{\Omega \cap \mathbb{M}_n^d} = F$?

Question 1.5. What groups can arise as the automorphism group of an NC domain?

A set $\Omega \subset \mathbb{M}^{[d]}$ is called *NC bounded* if for each n there exists a constant M_n such that

$$\forall z \in \Omega_n, \quad \|z\| < M_n.$$

In theorem 3.3 we show that one always has rigidity on NC-bounded domains that contain the origin. For such domains, the possible automorphism groups are therefore no more than certain subgroups of the automorphism groups of bounded domains (and we substantially answer questions 1.3–1.5 for circular bounded domains that contain the origin). In example 4.7 we show that many different domains can have the same automorphism groups.

In § 4 we consider the Cartan domain of type I, the set R_{pq} of $p \times q$ contractive matrices. The obvious NC domain containing this, where numbers are replaced by $n \times n$ matrices, we call $\mathcal{R}_{pq}$. In theorem 4.4 we show that when $p \neq q$ all automorphisms of R_{pq} extend, but when $p = q$ only those automorphisms that do not involve the transpose extend.

In this paper we restrict our attention to NC bounded domains, as the unbounded case is much more complicated (see, for instance, example 3.5).

This paper continues work of Popescu [16–18] and of Helton *et al.* [10].

2. Background on NC functions

The recent monograph [13] by Kaliuzhnyi-Verbovetskyi and Vinnikov gives an introduction to NC functions. Unless an additional hypothesis of continuity (or boundedness) is added, NC functions can behave badly.

EXAMPLE 2.1. Let $d = 1$ and define a function f on Jordan blocks by sending a Jordan block with 0 eigenvalues to the zero matrix of the same size, and a Jordan block with non-zero eigenvalues to the identity matrix of that size. Extend f by direct sums to any matrix in Jordan canonical form, and then by similarity to any matrix. The function f is then an NC function that is manifestly discontinuous.

Let σ denote the disjoint union topology on $\mathbb{M}^{[d]}$: a set U is in σ if and only if $U \cap \mathbb{M}_n^d$ is open for every n. (This topology is called the finitely open topology in [13].)

It was proved in [1] that if an NC function f on an NC set Ω is σ locally bounded, in the sense that

$$\forall z \in \Omega \; \exists \, U \in \sigma \text{ such that } z \in U \text{ and } f|_{\Omega \cap U} \text{ is bounded,}$$

then $f\colon \Omega \to \mathbb{M}^{[1]}$ is σ-σ continuous, and in [11] it was shown that this in turn implied that f was holomorphic, in the sense that

$$\forall n \in \mathbb{N}, \; \forall z \in \Omega_n, \; \forall h \in \mathbb{M}_n^d \; \exists \, Df(z)[h] = \lim_{t \to 0} \frac{1}{t}[f(z+th) - f(z)]. \tag{2.1}$$

Putting these results together, we come to the following proposition.

PROPOSITION 2.2. *An NC map f into an NC-bounded domain is automatically σ-σ continuous, and holomorphic in the sense of (2.1).*

3. Rigidity

A domain is called *circular* if it is invariant under multiplication by unimodular scalars.

The following lemmas are classical and due to H. Cartan.

LEMMA 3.1 (H. Cartan [7, théorème VII, p. 30]). *Let $\mathcal{D} \subseteq \mathbb{C}^d$ be a bounded domain, let $z_0 \in \mathcal{D}$ and let $\phi\colon \mathcal{D} \to \mathcal{D}$ be a biholomorphic automorphism with $\phi(z_0) = z_0$ and $\phi'(z_0) = I_n$. Then ϕ is the identity.*

LEMMA 3.2 (H. Cartan [6], [7, théorème VI]). *If $\mathcal{D}$ is a bounded circular domain in $\mathbb{C}^d$ containing 0, and $F\colon \mathcal{D} \to \mathcal{D}$ is a biholomorphic automorphism of $\mathcal{D}$ with $F(0) = 0$, then F is the restriction to $\mathcal{D}$ of an invertible linear map.*

THEOREM 3.3. *Let Ω be an NC domain that is NC bounded. Let $\Phi = (\Phi^1, \ldots, \Phi^d)$ be an NC automorphism of Ω. Suppose that for some $m \in \mathbb{N}$, we have $0 \in \Omega \cap \mathbb{M}_m^d$ and $(\Phi|_{\Omega_m})(0) = 0$.*

(i) *If, in addition, $(\Phi|_{\Omega_m})'(0)$ is the identity, then Φ is the identity on all of Ω.*

(ii) *If instead we suppose also that Ω is a circular NC domain, then there is an invertible linear map F on $\mathbb{C}^d$ such that $\Phi(Z) = (F \otimes \mathrm{id}_n)(Z)$ for $Z \in \Omega_n$ (by which we mean that each d-tuple $(\Phi(Z))_{i,j}$ formed from the (i,j) coordinates of the d-tuple of $n\times n$ matrices $\Phi(Z)$ is given by $F(Z_{i,j})$, where $Z = (Z^1, \ldots, Z^d)$ and $Z_{i,j} = (Z^1_{i,j}, \ldots, Z^d_{i,j})$ again denotes the (i,j) coordinates).*

Proof. (i) By lemma 3.1, $\Phi|_{\Omega_m}$ is the identity.

As $\Omega_m = \Omega \cap \mathbb{M}_m^d$ is open, there is some $\varepsilon > 0$ such that if $z = (z^1, \ldots, z^d) \in \mathbb{M}_m^d$ has each $\|z^j\| < \varepsilon$, then $z \in \Omega_m$.

Now, fix a positive integer k. For $Z^j \in \mathbb{M}_{km}$, we write $\hat{Z}^j$ for the d-tuple $(Z^1, \ldots, Z^d) \in \mathbb{M}_{km}^d$ that has $Z^i = 0$ when $i \neq j$, and whose jth entry is Z^j. If Z^j is the direct sum of k matrices from $\mathbb{M}_m$, and if $\|Z^j\| \leqslant \varepsilon$, then $\Phi(\hat{Z}^j) = \hat{Z}^j$, by the direct sum property of NC maps. As this applies to ζZ^j for $|\zeta| < 1$ ($\zeta \in \mathbb{C}$), it follows that the directional derivative satisfies $((\Phi|_{\Omega_{km}})'(0))(\hat{Z}^j) = \hat{Z}^j$. By linearity of the Fréchet derivative $(\Phi|_{\Omega_{km}})'(0)$, we may drop the restriction that $\|Z^j\| < \varepsilon$. In particular, the conclusion holds when $Z^j \in \mathbb{M}_{km}$ is a diagonal matrix.

By the chain rule and similarity invariance

$$(\Phi|_{\Omega_{km}})(s^{-1}Zs) = s^{-1}(\Phi|_{\Omega_{km}})(Z)s$$

of the map $\Phi|_{\Omega_{km}}$ (valid for all sufficiently small Z once s is fixed), we must have that $(\Phi|_{\Omega_{km}})'(0)$ has the invariance property

$$((\Phi|_{\Omega_{km}})'(0))(s^{-1}Zs) = s^{-1}(((\Phi|_{\Omega_{km}})'(0))(Z))s. \tag{3.1}$$

Choosing Z^j diagonalizable and s such that $s^{-1}Z^js$ is diagonal, (3.1) yields

$$s^{-1}(((\Phi|_{\Omega_{km}})'(0))(\hat{Z}^j))s = s^{-1}\hat{Z}^js,$$

and hence $((\Phi|_{\Omega_{km}})'(0))(\hat{Z}^j) = \hat{Z}^j$ provided that Z^j is diagonalizable. By density of the diagonalizable matrices we can then make the same conclusion with Z^j arbitrary, and then linearity of $(\Phi|_{\Omega_{km}})'(0)$ forces it to be the identity.

By lemma 3.1 again, $\Phi|_{\Omega_{km}}$ must be the identity.

Now choose some n, not necessarily a multiple of m, such that Ω_n is non-empty. Let $Z \in \Omega_n$. The direct sum of m copies of Z is in Ω_{mn}, and $\Phi(\bigoplus_{i=1}^m Z) = \bigoplus_{i=1}^m Z$ by the first part of the proof. As Φ preserves direct sums, this means that $\Phi(Z) = Z$.

(ii) By lemma 3.2, we know that $\Phi|_{\Omega_m}$ is linear. If $m = 1$, we take F to be $\Phi|_{\Omega_m}$. For $m > 1$ we need a brief argument to find F.

Similarity invariance

$$(\Phi|_{\Omega_m})(s^{-1}Zs) = s^{-1}(\Phi|_{\Omega_m})(Z)s$$

(guaranteed by definition 1.2 to hold for $Z, s^{-1}Zs \in \Omega_m$) must hold globally for $Z \in \mathbb{M}_m^d$ in view of linearity. Let $E_{i,k}$ denote the standard matrix units in $\mathbb{M}_m$ and choose $Z = (z^1E_{1,1}, \ldots, z^dE_{1,1})$ (so that Z is supported on the $(1,1)$ entries). Consider a block diagonal $s = 1 \oplus t$ with 1 in the $(1,1)$ entry but arbitrary invertible $(m-1)\times(m-1)$ block t. Since the matrices that commute with all such $1 \oplus t$ are those of the form $F \oplus \alpha I_{m-1}$ (for scalars F and α) we see that $\Phi|_{\Omega_m}(Z)$ must take the form

$$(F^1(z) \oplus \alpha^1(z)I_{m-1}, \ldots, F^d(z) \oplus \alpha^d(z)I_{m-1}) \quad \text{with } z = (z^1, \ldots, z^d)$$

for some scalar-valued linear functions $F^1, \ldots, F^d, \alpha^1, \ldots, \alpha^d \colon \mathbb{C}^d \to \mathbb{C}$.

However, notice that (by lemma 3.2) we also know that $\Phi|_{\Omega_{2m}}$ is linear, and by the direct sum property $\Phi(Z \oplus 0) = \Phi(Z) \oplus 0$. The similarity argument applied to $\mathbb{M}_{2m}^d$ forces $\alpha^j = 0$ ($1 \leqslant j \leqslant d$). Thus,

$$\Phi|_{\Omega_m}(z^1 E_{1,1}, \ldots, z^d E_{1,1}) = (F^1(z)E_{1,1}, \ldots, F^d(z)E_{1,1})$$

(with $z = (z^1, \ldots, z^d)$). Taking s to be a transposition, and using similarity, we conclude that the same must hold for $E_{k,k}$ replacing $E_{1,1}$. Taking $s = I_n - E_{i,k}$ for $i \neq k$ we have $s^{-1}E_{ii}s = E_{ii} - E_{ik}$, and together with linearity we deduce the relation with $E_{k,k}$ replaced by $E_{i,k}$. Clearly, $F = (F^1, \ldots, F^d)\colon \mathbb{C}^d \to \mathbb{C}^d$ must be invertible (since Φ is) and we have the desired conclusion on Ω_m.

By lemma 3.2, we know that $\Phi|_{\Omega_{km}}$ is linear for each $k \in \mathbb{N}$. For $Z^j \in \mathbb{M}_{km}$, if Z^j is diagonal, we must have $\Phi(\hat{Z}^j)$ of the required form. The similarity property and density of the diagonalizable matrices in $\mathbb{M}_{km}$ allows us to extend to arbitrary non-zero Z^j. Then by linearity this extends to arbitrary d-tuples $Z = (Z^1, \ldots, Z^d) \in \mathbb{M}_{km}^d$.

Finally, if Ω_n is non-empty for some n, we can apply the result just obtained for Ω_{nm} together with the direct sum property for $Z \oplus 0 \oplus \cdots \oplus 0$ (where $Z \in \mathbb{M}_n$ and we have $(m-1)$ zero summands) to establish the desired conclusion for $\Phi|_{\Omega_n}$. □

Popescu's Cartan uniqueness results [17, § 1] can be viewed as similar in spirit for the case of special domains (row contractions) to theorem 3.3. In the $m = 1$ case, the result of [11, corollary 4.1(2)] is part (i), while [12, theorem 21] implies (ii).

Example 3.4 (the matrix polydisc). This is the set

$$\mathcal{D} = \{x \in \mathbb{M}^{[d]}\colon \|x^j\| < 1,\ 1 \leqslant j \leqslant d\}.$$

The set of automorphisms of $\mathcal{D}$ is the set

$$\{\Phi(x) = \sigma \circ (m^1(x^1), \ldots, m^d(x^d))\colon \sigma \in \mathcal{S}_d,\ m^j \in \mathrm{Aut}(\mathbb{D})\}. \tag{3.2}$$

Here, $\mathcal{S}_d$ is the symmetric group on d variables, and $\mathrm{Aut}(\mathbb{D})$ is the Möbius group of automorphisms of the disk. Each Möbius transformation of the form

$$m(z) = \mathrm{e}^{\mathrm{i}\theta}\frac{z - a}{1 - \bar{a}z}$$

extends to matrices in the obvious way:

$$m(Z) = \mathrm{e}^{\mathrm{i}\theta}(Z - aI)(1 - \bar{a}Z)^{-1}.$$

Indeed, by von Neumann's inequality, every Φ in (3.2) extends to an automorphism of $\mathcal{D}$. That this comprises everything follows from observing that all automorphisms of the polydisc $\mathbb{D}^d = \mathcal{D} \cap \mathbb{M}_1^d$ are of this form, and so by theorem 3.3 they have a unique extension to higher levels. As they are invertible, they must be automorphisms.

Example 3.5. Theorem 3.3 fails if boundedness is dropped. Consider, for example, the NC set

$$\Omega = \{(x, y, z) \in \mathbb{M}^{[3]}\colon \|xy - yx\| < 1\}.$$

Let

$$\Phi(x, y, z) = (x, y, z + h(xy - yx)),$$

where $h\colon \mathbb{C} \to \mathbb{C}$ is any non-constant entire function mapping 0 to 0. Then Φ is an automorphism, and $\Phi|_{\Omega\cap\mathbb{M}_1^3}$ is the identity, but Φ is not the identity on level 2.

4. Extendibility in $\mathcal{R}_{pq}$

Let R_{pq} denote the $p \times q$ matrices of norm less than 1. We shall extend this to an NC domain in $\mathbb{M}^{[d]}$, where $d = pq$, by

$$\mathcal{R}_{pq} := \bigcup_{n=1}^{\infty} \left\{ (x^1, x^2, \dots, x^d) \in \mathbb{M}_n^d \colon \left\| \begin{pmatrix} x^1 & \cdots & x^q \\ x^{q+1} & \cdots & x^{2q} \\ \vdots & \ddots & \vdots \\ x^{(p-1)q+1} & \cdots & x^{pq} \end{pmatrix} \right\| < 1 \right\}.$$

Let $\gamma\colon \mathbb{M}_n^d \to \mathbb{M}_{pq}^d$ be the map that takes d matrices and makes them into a block $p\times q$ matrix, filling in left to right and then top to bottom. Then $\mathcal{R}_{pq} = \{x\colon \|\gamma(x)\| < 1\}$. When we speak of an NC automorphism of $\mathcal{R}_{pq}$, strictly speaking we mean an NC automorphism of $\gamma^{-1}(\mathcal{R}_{pq})$.

In the special case in which $q = 1$, R_{p1} is just the unit ball in $\mathbb{C}^p$, and its automorphisms are well known (see, for example, [20, theorem 2.2.5]). The set $\mathcal{R}_{1q}$, the row contractions, was studied by Popescu in [15–17].

The automorphisms of R_{pq} are given by a similar formula to the case of the ball. Harris showed [9] that they are of the following form.

THEOREM 4.1 (Harris [9]). *Every holomorphic automorphism of R_{pq} is of the form LH_A where L is a linear isometric automorphism of R_{pq}, A is an element of R_{pq}, and*

$$H_A(x) = (I_p - AA^*)^{-1/2}(x + A)(I_q + A^*x)^{-1}(I_q - A^*A)^{1/2}.$$

First, let us consider automorphisms that map 0 to 0, which are the linear ones. Morita [14] classified the linear isometries of R_{pq}, and the square case differs from the rectangular case because the transpose is an isometry.

THEOREM 4.2 (Morita [14]). *If $p \neq q$, all linear automorphisms of R_{pq} are of the form $x \mapsto UxV$, where U is a $p \times p$ unitary and V is a $q \times q$ unitary. If $p = q$, the set of linear automorphisms consists of $x \mapsto UxV$ and $x \mapsto Ux^{\mathrm{T}}V$.*

The map $x \mapsto UxV$ extends to the NC automorphism of $\mathcal{R}_{pq}$ given by $Z \mapsto (\mathrm{id} \otimes U)Z(\mathrm{id} \otimes V)$, but the transpose does not extend.

LEMMA 4.3. *If $p > 1$, the map $x \mapsto x^{\mathrm{T}}$ does not extend to an NC automorphism of $\mathcal{R}_{pp}$.*

Proof. Using theorem 3.3, if the transpose did extend, the extension would map $(X_{i,j}) \in \mathbb{M}_p(\mathbb{M}_n(\mathbb{C}))$ to $(X_{j,i})$, and so this reduces to the well-known fact that the transpose map is not a complete isometry of $\mathbb{M}_p$. □

Let us turn now to H_A. The map H_A extends to an NC map from $\mathcal{R}_{pq}$ to $\mathcal{R}_{pq}$ given by

$$H_A(Z) = (I_{n,p} - \mathrm{id} \otimes AA^*)^{-1/2}(Z + \mathrm{id} \otimes A)(I_{n,q} + (\mathrm{id} \otimes A^*)Z)^{-1}(I_{n,q} - \mathrm{id} \otimes A^*A)^{1/2}. \tag{4.1}$$

Here, id means $\mathrm{id}_{\mathbb{C}^n}$, and $I_{n,r}$ denotes $\mathrm{id}_{\mathbb{C}^n \otimes \mathbb{C}^r}$.

A calculation shows that

$$\begin{aligned}\mathrm{id} - H_A(W)^* H_A(Z) = (I_{n,q} - \mathrm{id} \otimes A^*A)^{1/2}(I_{n,q} + W^*(\mathrm{id} \otimes A))^{-1} \\ \times (I_{n,q} - W^*Z)(I_{n,q} + (\mathrm{id} \otimes A^*)Z)^{-1} \\ \times (I_{n,q} - \mathrm{id} \otimes A^*A)^{1/2}.\end{aligned} \tag{4.2}$$

Letting $W = Z$ proves that H_A maps $\mathcal{R}_{pq}$ to $\mathcal{R}_{pq}$ and, as H_{-A} is the inverse of H_A, it must be an automorphism.

Putting these results together, we get the following theorem. The $p = 1$ case was proved by Popescu [17, theorem 1.5]. The general case was proved by Helton *et al.* [10, theorem 1.7], though their hypotheses are stronger. The linear case was proved by Blecher and Hay [5].

THEOREM 4.4. *If $p \neq q$, then every automorphism of R_{pq} extends uniquely to an automorphism of $\mathcal{R}_{pq}$. If $p = q$, the automorphisms of the form $x \mapsto UH_A(x)V$ extend uniquely to $\mathcal{R}_{pp}$, and the automorphisms of the form $x \mapsto UH_A(x)^{\mathrm{T}}V$ (when $p > 1$) do not extend to automorphisms of $\mathcal{R}_{pp}$.*

By a result of Ball and Bolotnikov [4] (see also [2,3]), H_A extends to an endomorphism of the *commuting* elements of $\gamma^{-1}(\mathcal{R}_{pq})$ if and only if there is some function F such that

$$I - H_A(w)^* H_A(z) = F(w)^*(I - w^*z)F(z)$$

as a kernel on R_{pq}. This is true, as (4.2) shows. So the NC automorphisms of $\{x \in \gamma^{-1}(\mathcal{R}_{pq})\colon x^i x^j = x^j x^i \text{ for all } 1 \leqslant i, j \leqslant d\}$ are the same as the NC automorphisms of $\gamma^{-1}(\mathcal{R}_{pq})$. This phenomenon has also been explored in [1].

QUESTION 4.5. The automorphisms of $\mathcal{R}_{pq}$ are not transitive at any level $n \geqslant 2$. What can one say about the orbits?

Theorem 4.4 can be extended slightly. For S a subset of $\mathbb{N}$ that is closed under addition, let $\mathcal{R}_{pq}(S)$ be defined as follows: $\mathcal{R}_{pq}(S) \cap \mathbb{M}_n^d$ is $\mathcal{R}_{pq} \cap \mathbb{M}_n^d$ if $n \in S$, and empty otherwise.

PROPOSITION 4.6. *Let S be any non-empty sub-semigroup of $\mathbb{N}$. Then the automorphisms of $\mathcal{R}_{pq}(S)$ are the same as the automorphisms of $\mathcal{R}_{pq}$, and are uniquely determined by their action on any non-empty level.*

EXAMPLE 4.7. Extendibility can fail if the pieces of Ω at different levels are not somehow alike. For example, let $d = 1$ and let $R > 1$. Define Ω by $\Omega \cap \mathbb{M}_1 = \mathbb{D}$, and $\Omega \cap \mathbb{M}_n = \{x\colon \|x\| < R\}$. The automorphisms of $\Omega \cap \mathbb{M}_1$ are the Möbius maps, but only multiplication by $\mathrm{e}^{\mathrm{i}\theta}$ extends to be an automorphism of Ω.

But there are many other choices of NC domain $\Omega \supset \mathcal{R}_{11}$ that have the same automorphism group. For example, let $r_1 = 1$ and let $(r_n)_{n=1}^{\infty}$ be any non-decreasing sequence. Define Ω by

$$\Omega_n = \{x \in \mathbb{M}_n\colon \exists s \in \mathbb{M}_n \text{ with } \|s\|\|s^{-1}\| \leqslant r_n \text{ and } \|s^{-1}xs\| < 1\}. \tag{4.3}$$

Then Ω is an NC domain and its automorphism group is the set of Möbius maps.

QUESTION 4.8. Let $U \subset \mathbb{C}^d$ be a bounded symmetric domain. Is there an NC domain $\Omega \subset \mathbb{M}^{[d]}$ such that $\Omega_1 = U$ and such that the automorphism group of Ω equals the automorphism group of U?

Note added in proof. In the recent preprint [19], Popescu studies automorphisms of a special class of NC-bounded circular NC domains, which he calls non-commutative polyballs. These are domains of the form $\mathcal{R}_{1q_1} \times \cdots \times \mathcal{R}_{1q_m}$, with the restriction that elements from distinct factors commute. For these domains, he proves among other things a version of theorem 3.3, and characterizes all their automorphisms.

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References

1 J. Agler and J. E. McCarthy. Global holomorphic functions in several non-commuting variables. *Can. J. Math.* **67** (2015), 241–285.

2 J. Agler, J. E. McCarthy and N. J. Young. On the representation of holomorphic functions on polyhedra. *Michigan Math. J.* **62** (2013), 675–689.

3 C.-G. Ambrozie and D. Timotin. A von Neumann type inequality for certain domains in $\mathbb{C}^n$. *Proc. Am. Math. Soc.* **131** (2003), 859–869.

4 J. A. Ball and V. Bolotnikov. Realization and interpolation for Schur–Agler class functions on domains with matrix polynomial defining function in $\mathbb{C}^n$. *J. Funct. Analysis* **213** (2004), 45–87.

5 D. P. Blecher and D. M. Hay. Complete isometries – an illustration of noncommutative functional analysis. In *Function spaces*, Contemporary Mathematics, vol. 328, pp. 85–97 (Providence, RI: American Mathematical Society, 2003).

6 H. Cartan. Les transformations analytiques des domaines cerclés les uns dans les autres. *C. R. Acad. Sci. Paris* **190** (1930), 718–720.

7 H. Cartan. Les fonctions de deux variables complexes et le problème de la représentation analytiques. *J. Math. Pures Appl.* **9** (1931), 1–114.

8 E. Cartan. Sur les domaines bornés homogènes de l'éspace de variables complexes. *Abh. Math. Sem. Univ. Hamburg* **11** (1935), 116–162.

9 L. A. Harris. Bounded symmetric homogeneous domains in infinite dimensional spaces. In *Proceedings on infinite dimensional holomorphy*, Lecture Notes in Mathematics, vol. 364, pp. 13–40 (Springer, 1974).

10 J. W. Helton, I. Klep, S. McCullough and N. Slinglend. Noncommutative ball maps. *J. Funct. Analysis* **257** (2009), 47–87.

11 J. W. Helton, I. Klep and S. McCullough. Proper analytic free maps. *J. Funct. Analysis* **260** (2011), 1476–1490.

12 J. W. Helton, I. Klep and S. McCullough. Free analysis, convexity and LMI domains. In *Mathematical methods in systems, optimization and control*, Operator Theory: Advances and Applications, vol. 222, pp. 195–219 (Springer, 2012).

13 D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov. *Foundations of free non-commutative function theory* (Providence, RI: American Mathematical Society, 2014).

14 K. Morita. Analytical characterization of displacements in general Poincaré space. *Proc. Imp. Acad. Tokyo* **17** (1941), 489–494.

15 G. Popescu. Von Neumann inequality for $(B(\mathcal{H})^n)_1$. *Math. Scand.* **68** (1991), 292–304.

16 G. Popescu. Free holomorphic functions on the unit ball of $B(\mathcal{H})^n$. *J. Funct. Analysis* **241** (2006), 268–333.

17 G. Popescu. Free holomorphic automorphisms of the unit ball of $B(\mathcal{H})^n$. *J. Reine Angew. Math.* **638** (2010), 119–168.

18 G. Popescu. Free biholomorphic classification of noncommutative domains. *Int. Math. Res. Not.* no. 4 (2011), 784–850.

19 G. Popescu. Holomorphic automorphisms of noncommutative polyballs. Preprint, 2015. (Available at http://arxiv.org/abs/1502.07905.)

20 W. Rudin. *Function theory in the unit ball of* C^n (Springer, 1980).

21 J. L. Taylor. The analytic functional calculus for several commuting operators. *Acta Math.* **125** (1970), 1–38.