Recent results on elementary operators

Richard M. Timoney

Trinity College Dublin

Elementary operators

A a C*-algebra

elementary operator $T\colon A \to A$ is

$$Tx = \sum_{i=1}^{\ell} a_i x b_i$$

$u = \sum_{i=1}^{\ell} a_i \otimes b_i \in M(A) \otimes M(A)$, $\qquad T = \theta(u)$

Special and general examples

Special: $L_a x = ax$, $R_b x = xb$, $\delta_a x = ax - xa$,
$\delta_{a,b} = L_a - R_b$,
$J_{a,b} x = (L_a R_b + R_a L_b)x = axb + bxa$
(ℓ small)

General: $\forall T \in \mathcal{E}\ell(A), \forall I$ ideal of A, $T(I) \subset I$.

$T\colon \mathcal{K}(H) \to \mathcal{K}(H)$ bounded linear $\Rightarrow T$ is in s.o.t. closure of $\mathcal{E}\ell(\mathcal{K}(H))$.

Haagerup norm estimate

$Tx = \sum_{i=1}^{\ell} a_i x b_i$

$$Tx = [a_1, a_2, \ldots, a_\ell](x \otimes I_\ell) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_\ell \end{bmatrix} = \mathbf{a}(x \otimes I_\ell)\mathbf{b}$$

$$\|T\| \leq \inf \|\mathbf{a}\| \|\mathbf{b}\| = \inf \frac{1}{2} \left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \right)$$

$$\|u\|_h \stackrel{def}{=} \inf\left\{\|\mathbf{a}\|\|\mathbf{b}\| : u = \sum_{i=1}^{\ell} a_i \otimes b_i\right\}$$

We have $\|\theta(u)\| \leq \|u\|_h$

$T^{(2)} : M_2(A) \to M_2(A)$ is

$$T^{(2)} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} Tx_{11} & Tx_{12} \\ Tx_{21} & Tx_{22} \end{pmatrix}$$

$$T^{(2)}(X) = \sum_{i=1}^{\ell} (a_i \otimes I_2) X (b_i \otimes I_2)$$

Same estimate $\|T^{(2)}\| \leq \|u\|_h$.

In general $\|T^{(k)}\| \leq \|u\|_h (\forall k = 2, 3, \ldots) \Rightarrow$
$\|T\|_{cb} \stackrel{def}{=} \sup_k \|T^{(k)}\| = \|\theta(u)\|_{cb} \leq \|u\|_h$
Good — at least for $A = \mathcal{B}(H)$ equalty holds

In general $\|\theta(u)\|_{cb} = \|u\|_h \forall u \in A \otimes A \iff A$ is a prime C^*-algebra (Mathieu)

Can use the $\mathcal{B}(H)$ case as a way to compute $\|u\|_h$ for $u \in A \otimes A$ — find a faithful representation $\pi\colon A \to \mathcal{B}(H_\pi)$ and try to compute $\|T\|_{cb}$ on $\mathcal{B}(H)$ where $T = \theta((\pi \otimes \pi)(u))$.

Stampfli (1970): $u = a \otimes 1 - 1 \otimes b$, $\theta(u) = \delta_{a,b}$. On $\mathcal{B}(H)$, $\|\delta_{a,b}\| = \inf_{\lambda \in \mathbb{C}} \|a - \lambda\| + \|b - \lambda\|$

His proof used numerical range ideas to recognise equality in the estimate.

Stampfli techniques generalised to arbitary $u \in \mathcal{B}(H) \otimes \mathcal{B}(H)$ and equality in $\|\theta(u)\|_{cb} \leq \|u\|_h$ (Illinois J. 2003)

Used joint numerical ranges of $a_i a_j^*$ and of $b_i^* b_j$.
$\{(\langle a_i a_j^* \xi, \xi \rangle)_{i,j=1}^{\ell} : \xi \in H, \|\xi\| = 1\}$

Theorem $T \in \mathcal{E}\ell(A)$, $Tx = \sum_{i=1}^{\ell} a_i x b_i$

Then $\|T\|_k = \|T\|_{cb}$ for $k \geq \ell$

Proof. uses a generalisation of the Toeplitz-Hausdorff theorem to joint numerical ranges.

Central Haagerup tensor norm

'Obvious' extension of $\|\theta(u)\|_{cb} \leq \|u\|_h$ ($u \in A \otimes A$) notes $\theta((az) \otimes b - a \otimes (zb)) = 0$. $A \otimes_{Z,h} A =$ quotient of $A \otimes_h A$ by the closed span of $\{(az) \otimes b - a \otimes (zb)\}$.

$$\theta_Z \colon A \otimes_{Z,h} A \to CB(A)$$

contraction: $\|\theta_Z(u)\|_{cb} \leq \|u\|_{Z,h}$.

Often isometric.

Primal ideals

An ideal $Q \subset A$ is *n-primal* if $J_1, J_2, \ldots, J_n \subset A$ ideals,

$$J_1 J_2 \ldots J_n = 0 \Rightarrow J_k \subset Q \text{ some } k$$

Q is *primal* if it is n-primal for each n.
Somerset (JOT '98)
$\|\theta(u)\|_{cb} = \sup\{\|u^Q\|_h : Q \subset A \text{ primal}\}$
where $u^Q = (\pi_Q \otimes \pi_Q)(u) \in (A/Q) \otimes (A/Q)$
and $\pi_Q \colon A \to A/Q$ is the quotient map.

Glimm ideals

A unital for now.

$M \subset Z$ maximal ideal of centre. $G = AM$ ideal of A generated by M is called a Glimm ideal of A.
Somerset (JOT '98)

$$\|u\|_{Z,h} = \sup\{\|u^G\|_h : G \in \text{ Glimm}(A)\}$$

Isometry of θ_Z

Theorem (A, S & T) For A unital, θ_Z is isometric $\iff$ each $G \in$ Glimm(A) is primal.

Somerset established $\Leftarrow$ part and showed that each G must be 3-primal (if θ_Z is isometric on derivations $u = a \otimes 1 - 1 \otimes a$).

Proof of $\Rightarrow$ requires a construction of u. And example of A to show G 3-primal $\forall G \in$ Glimm$(A) \not\Rightarrow G$ primal $\forall G \in$ Glimm(A).

Step 0: G is n-primal $\iff$ whenever $G \subset P_j$ and $P_j \in \mathrm{Prim}(A) \forall j (1 \le j \le n)$ then $Q = \bigcap_j P_j$ primal.

Step 1: $\exists G \in \mathrm{Glimm}(A)$ not primal $\Rightarrow G$ not n-primal but G is $(n-1)$-primal (some $n \ge 2$). Hence $\exists P_j \in \mathrm{Prim}(A)$ $(1 \le j \le n)$ such that $G \subset P_j \forall j$, $I = \bigcap_j P_j$ not primal but $R_j = \bigcap_{k \ne j} P_k$ is primal $\forall j$.

Step 2: Find $b_1, b_2, \ldots, b_n \in A$ orthog., positive norm 1, $\|b_j + G\| = 1 \forall j$ but $P \in \mathrm{Prim}(A) \Rightarrow \exists k, b_k \in P$.

End: $u = (\sum_j b_j) \otimes (\sum_j b_j) - \sum_j b_j \otimes b_j$, $\|u\|_h = 2\left(1 - \frac{1}{n}\right)$

Theorem

A still unital, $\ell > 0$ now fixed. $u = \sum_{i=1}^{\ell} a_i \otimes b_i$ arbitrary (of length $\leq \ell$).

$\|\theta_Z(u)\|_{cb} = \|u\|_{Z,h}$ for all such $u \iff$ each $G \in \mathrm{Glimm}(A)$ is N-primal for $N = \ell^2 + 1$.

Formula for $\|T\|$

For $X, Y \in M_\ell^+$, define the tracial geometric mean
tgm$(X, Y) =$ trace$((X^{1/2} Y X^{1/2})^{1/2})$.
For $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^t$, $\eta \in H$, $\|\eta\| = 1$, let

$$Q(\mathbf{b}, \eta) = (\langle b_i^* b_j \eta, \eta\rangle)_{i,j=1}^{\ell}$$

For $T \in \mathcal{E}\ell(\mathcal{B}(H))$, $Tx = \sum_{i=1}^{\ell} a_i x b_i$,

$$\|T\| = \sup_{\xi,\eta} \mathrm{tgm}\left(Q(\mathbf{a}^*, \xi), Q(\mathbf{b}, \eta)\right)$$

Corollary (A, M, S using Magajna) If $T \in \mathcal{E}\ell(A)$ and A antiliminal, then $\|T\|_{cb} = \|T\|$.

Corollary $T \in \mathcal{E}\ell(A)$,
$Tx = \sum_{j=1}^{\ell} a_j x b_j \Rightarrow \|T\|_{cb} \leq \sqrt{\ell}\|T\|$.

Theorem $T \in \mathcal{E}\ell(A)$, T compact $\Rightarrow Tx = \sum_{j=1}^{\ell} a_j x b_j$ with $a_j, b_j \in \mathcal{K}(A)$.