MA342A (Harmonic Analysis 1) Tutorial/exercise sheet 8
[due December 13, 2017]

Name: Solutions

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1. Let $F \in L^2(\mathbb{T})$. For $N \in \mathbb{N}$, show

$$\sum_{n=-N}^{N} |\hat{F}(n)|^2 \leq \|F\|_2^2.$$

Deduce that

$$\sum_{n=-\infty}^{\infty} |\hat{F}(n)|^2 \leq \|F\|_2^2.$$

[Hint: see the proof for $F \in C(\mathbb{T})$.]

Solution: The proof in the notes of Bessel's inequality (Proposition 2.1.2) was given for $F \in C(\mathbb{T})$ but the same proof works for $F \in L^2(\mathbb{T})$.

2. Could there be $F \in L^2(\mathbb{T})$ such that $\hat{F}(n) = 1/\sqrt{n}$ for $n \geq 1$ and $\hat{F}(n) = 0$ for $n \leq 0$?

Solution: No because if there was such F, then

$$\sum_{n=-\infty}^{\infty} |\hat{F}(n)|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{n} = \infty$$

and that would violate the second inequality in Q1.

3. Could there be $F \in L^1(\mathbb{T})$ such that $\hat{F}(n) = (-1)^n$ for each $n \in \mathbb{Z}$?

Solution: No because that would violate the Riemann Lebesgue Lemma (Corollary 2.6.9 in the notes).

4. Show that there is $F \in C(\mathbb{T})$ with $\hat{F}(n) = 1/2^n$ for $n \geq 1$ and $\hat{F}(n) = 0$ for $n \leq 0$.

Solution: We consider the sum

$$F(\zeta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \zeta^n$$

for $\zeta \in \mathbb{T}$. As explained in the proof of Theorem 2.6.2 in the notes, this series is uniformly convergent for $\zeta \in \mathbb{T}$ and so defines $F \in C(\mathbb{T})$.

We could invoke the Weierstrass M-test to show this or use a bare hands proof as in the proof of Theorem 2.6.2. The Weierstrass M-test says that if $f_n \colon X \to \mathbb{C}$ are functions on a set X and there exist $M_n \geq 0$ such that $\sup_{x \in X} |f_n(x)| \leq M_n$ for each n and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on X (meaning that $f(x) =$

$\lim_{N\to\infty} \sum_{n=1}^N f_n(x)$ exists for each $x \in X$ and the limit is uniform in the sense that given $\varepsilon > 0$ there is N_0 such that

$$N \geq N_0 \Rightarrow \left| f(x) - \sum_{n=1}^N f_n(x) \right| < \varepsilon \quad (\forall x \in X).$$

(The uniformity is that the the same N_0 works for all $x \in X$ at the same time.) One usual consequence of the M-test if that if we assume X is a topological space (as $X = \mathbb{T}$ is in our problmem) and that each f_n is continuous, then f must be continuous as a uniform limit of continuous functions. Also, if it makes sense to integrate over X we can integrate f by exchanging the order of the sum and the integral.

Instead of using the M-test stuff, we could follow the model used in the proof of Theorem 2.6.2. For $\zeta \in \mathbb{T}$,

$$\sum_{n=1}^\infty \left| \frac{1}{2^n} \zeta^n \right| = \sum_{n=1}^\infty \frac{1}{2^n} = 1 < \infty$$

and so the series $\sum_{n=1}^\infty \frac{1}{2^n} \zeta^n$ is absolutely convergent, hence convergent. It makes sense then to define $F(\zeta)$ as we did above. To show uniform convergence, take $\varepsilon > 0$ and choose N_0 large enough that $\sum_{n=N_0+1}^\infty 1/2^n < \varepsilon$. Then for $N \geq N_0$ we have

$$\left| F(\zeta) - \sum_{n=1}^N \frac{1}{2^n} \zeta^n \right| = \left| \sum_{n=N+1}^\infty \frac{1}{2^n} \zeta^n \right| \leq \sum_{n=N+1}^\infty \frac{1}{2^n} \leq \sum_{n=N_0+1}^\infty \frac{1}{2^n} < \varepsilon.$$

That established uniform convergence of the partial sums $\sum_{n=1}^N \frac{1}{2^n} \zeta^n$ to $F(\zeta)$. As the partical sums are continuous on $\mathbb{T}$, we have $F \in C(\mathbb{T})$.

Finally we claim that $\hat{F}(n) = 1/2^n$ for $n \geq 0$ and $\hat{F}(n) = 0$ for $n \leq 0$.

$$\begin{aligned}
\hat{F}(n) &= \int_{\mathbb{T}} F(\zeta) \zeta^{-n} \, d\lambda(\zeta) \\
&= \int_{\mathbb{T}} \left(\lim_{N\to\infty} \sum_{m=1}^N \frac{1}{2^m} \zeta^m \right) \zeta^{-n} \, d\lambda(\zeta) \\
&= \int_{\mathbb{T}} \lim_{N\to\infty} \sum_{m=1}^N \frac{1}{2^m} \zeta^{m-n} \, d\lambda(\zeta) \\
&= \lim_{N\to\infty} \sum_{m=1}^N \frac{1}{2^m} \int_{\mathbb{T}} \zeta^{m-n} \, d\lambda(\zeta)
\end{aligned}$$

(where I am using λ for normalised arc length measure on $\mathbb{T}$). The last step is justified by uniform convergence of the series. We know $\int_{\mathbb{T}} \zeta^{m-n} \, d\lambda(\zeta) = 0$ if $m \neq n$ and is 1 if $m = n$. So if $n \leq 0$ we always get 0 but of $n \geq 1$ we pick up one nonzero contribution of $1/2^n$ once $N \geq n$. Hence the Fourier coefficients $\hat{F}(n)$ are as claimed.

5. For $F \in L^1(\mathbb{T})$, let $\sigma_N F = (1/N)(S_0 F + S_1 F + S_2 F + \cdots + S_{N-1} F)$. (Note[1].) Compute $V_N F = 2\sigma_{2N} F - \sigma_N F$ in terms of the Fourier coefficients of F.

   Show that $\lim_{N\to\infty} \|V_N F - F\|_1 = 0$ (for $F \in L^1(\mathbb{T})$).

   Solution: We know

$$S_N F(\zeta) = \sum_{n=-N}^{N} \hat{F}(n)\zeta^n, \quad \sigma_N F(\zeta) = \sum_{n=-N}^{N} \hat{(}1 - |n|/N)F(n)\zeta^n,$$

   and so

$$\begin{aligned} V_N F(\zeta) &= 2\sigma_{2N} F(\zeta) - \sigma_N F(\zeta) \\ &= 2\sum_{n=-2N}^{2N} \hat{(}1 - |n|/(2N))F(n)\zeta^n - \sum_{n=-N}^{N} \hat{(}1 - |n|/N)F(n)\zeta^n \\ &= \sum_{n=-N}^{N} \hat{(}2 - |n|/N - 1 + |n|/N)F(n)\zeta^n + \sum_{N<|n|<2N} 2(1 - |n|/(2N))F(n)\zeta^n \\ &= \sum_{n=-N}^{N} \hat{F}(n)\zeta^n + \sum_{N<|n|<2N} 2(1 - |n|/(2N))F(n)\zeta^n \end{aligned}$$

   We also know from Fejér's theorem (Corollay 2.6.1) applied to the case $B = L^1(\mathbb{T})$ that

$$\lim_{N\to\infty} \|F - \sigma_N F\|_1 = 0.$$

   It follows of course that $\lim_{N\to\infty} \|F - \sigma_{2N} F\|_1 = 0$ and so

$$\begin{aligned} \lim_{N\to\infty} \|V_N F - F\|_1 &= \lim_{N\to\infty} \|2\sigma_{2N} F - 2F - \sigma_N F + F\|_1 \\ &= \lim_{N\to\infty} \|2(\sigma_{2N} F - F) - (\sigma_N F - F)\|_1 \\ &\leq \lim_{N\to\infty} 2\|\sigma_{2N} F - F\|_1 + \|\sigma_N F - F\|_1 = 0 \end{aligned}$$

6. Find the Cesàro sum of $\sum_{n=0}^{\infty} z^n$ for $z \in \mathbb{C}$, $|z| < 1$.

   Solution: For $z \neq 1$ fixed, the partial sums are

$$s_n \sum_{j=0}^{n} z^j = \frac{1 - z^{n+1}}{(} 1 - z)$$

   (Geometric sum formula $a + ar + \cdots + ar^{n-1} = a(1 - r^n)(1 - r)$ valid if $r \neq 1$.) One could argue that s_0 is the first partial sum.

[1]There should be the $S_0 F$ term to make this consistent with the notation in the notes. 8/12/207

Then, the Cesàro sums are

$$\begin{aligned}\sigma_n &= \frac{1}{n}(s_0 + s_1 + \cdots + s_{n-1}) \\ &= \frac{1}{n}\sum_{j=0}^{n-1} \frac{1-z^j}{1-z} \\ &= \frac{1}{n(1-z)}\left(n - \frac{1-z^n}{1-z}\right) \\ &= \frac{1}{1-z} - \frac{1-z^n}{n(1-z)^2}\end{aligned}$$

If $|z| = 1$ (and we are already exclduing $z = 1$) the z^n term has modulus 1 and so the limit is $\lim_{n\to\infty} \sigma_n = 1/(1-z)$.

That is the Cesàro sum of the series (but it is not a convergent series because $z^n \not\to 0$ as $n \to \infty$.

If we look at the case $z = 1$, we find that $s_n = n+1$ and $\sigma_n = \frac{1}{n}\sum_{j=0}^{n-1} s_j = \frac{1}{n}\sum_{j=0}^{n-1} j+1 = \frac{1}{n}\sum_{k=1}^{n} k = \frac{1}{n}\frac{n(n+1)}{2} = (n+1)/2 \to \infty$.

(The formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ is a well-known one that can be established by induction. If you did not know it you could argue using the fact that $s_j > n/2$ for about half the values of j, those with $j > n/2$, and then that will give σ_n bigger than about $\frac{1}{n}(n/2)^2$. To make that watertight you would need to account for things a little more carefully, depending on n even or odd and stuff.)

7. Let $G = \mathbb{R}$. What is Haar measure $\lambda_{\mathbb{R}}$? Show that the characteristic function of the unit interval $[0, 1]$ is in $L^1(\mathbb{R})$ and compute its Fourier transform.

   Solution: Haar measure $\lambda_{\mathbb{R}}$ is usually taken to be just Lebesgue length measure (though the definition allows any positive multiple of that measure).

   The characteristic function $\chi_{[0,1]}$ is certainly in $L^1(\mathbb{R})$ because it is measurable and has integral (of its absolute value) $\int_{\mathbb{R}} |\chi_{[0,1]}|\, dx = 1$.

   It's Fourier transform is

$$\widehat{\chi_{[0,1]}}(\xi) = \int_{\mathbb{R}} \chi_{[0,1]}(x) e^{-2\pi i x\xi}\, dx = \int_0^1 e^{-2\pi i x\xi}\, dx = \left[\frac{e^{-2\pi i x\xi}}{-2\pi i\xi}\right]_{x=0}^1 = \frac{1 - e^{-2\pi i\xi}}{2\pi i\xi}$$

   (for $\xi \in \mathbb{R}$).

8. For $G = \mathbb{Z}$, what is Haar measure $\lambda_{\mathbb{Z}}$? What does it mean for $F, H \colon \mathbb{Z} \to \mathbb{C}$ to be equal almost everywhere? How is the Fourier transform of $F \in L^1(\mathbb{Z}, \lambda_{\mathbb{Z}})$ defined?

   Solution: Haar measure $\lambda_{\mathbb{Z}}$ is counting measure (usually taken as that).

   The only subset of $\mathbb{Z}$ with counting measure zero is the empty set. So $F(n) = H(n)$ almost everywhere means $\lambda_{\mathbb{Z}}(\{n : F(n) \neq H(n)\}) = 0$ or $F(n) = H(n)$ everywhere (for all $n \in \mathbb{Z}$).

The characters of $\mathbb{Z}$ are in one to one correspondence with elements $\zeta \in \mathbb{T}$ or all the characters are of the form $\chi_\zeta(n) = \zeta^n$, where $\zeta = \chi_\zeta(1)$. The Fourier transform of $F \in L^1(\mathbb{Z}, \lambda_{\mathbb{Z}})$ is

$$\hat{F}(\chi_\zeta) = \hat{F}(\zeta) = \int_{\mathbb{Z}} F(n)\overline{\zeta^n}\, d\lambda_{\mathbb{Z}}(n) = \sum_{n=-\infty}^{\infty} F(n)\bar{\zeta}^n$$

9. For $G = \mathbb{Z}$, find an exhaustion $\mathbb{Z} = \bigcup_{n=1}^\infty K_n$ of $\mathbb{Z}$ by compact sets.

   Solution: We could take $K_n = \{-n, -n+1, \ldots, n-1, n\} = \{j \in \mathbb{Z} : |j| \leq n\}$. The K_n are compact (since finite), have union $\mathbb{Z}$ and $K_n \subset K_{n+1}^\circ = K_{n+1}$ (as $\mathbb{Z}$ has the discrete topology where all subsets are open).

   For the distance function $d_n \colon \hat{\mathbb{Z}} \times \hat{\mathbb{Z}} \to [0, \infty)$ given by

$$d_n(\chi_1, \chi_2) = \sup_{g \in K_n} |\chi_1(g) - \chi_2(g)|$$

   find $\chi \in \hat{\mathbb{Z}}$ with $d_n(\chi, 1) = 0$ but $\chi \neq 1$. [Hint: $\hat{\mathbb{Z}}$ is given in Proposition 1.3.11 (iii). Here 1 stands for the trivial character, which could also be written χ_0, $\chi_0(n) = 1$ for all $n \in \mathbb{Z}$.]

   Solution: There is no such χ as soon as $1 \in K_n$ because if $1 \in K_n$, then $d_n(\chi, 1) = 0$ implies $|\chi(1) - 1| = 0 \Rightarrow \chi(1) = 1 \Rightarrow \chi(n) = \chi(1)^n = 1$ for all n. So $\chi = 1$.

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