MA342A (Harmonic Analysis 1) Tutorial/exercise sheet 8
[due December 13, 2017]

Name:

Student ID:

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1. Let $F \in L^2(\mathbb{T})$. For $N \in \mathbb{N}$, show
$$\sum_{n=-N}^{N} |\hat{F}(n)|^2 \leq \|F\|_2^2.$$
Deduce that
$$\sum_{n=-\infty}^{\infty} |\hat{F}(n)|^2 \leq \|F\|_2^2.$$
[Hint: see the proof for $F \in C(\mathbb{T})$.]

2. Could there be $F \in L^2(\mathbb{T})$ such that $\hat{F}(n) = 1/\sqrt{n}$ for $n \geq 1$ and $\hat{F}(n) = 0$ for $n \leq 0$?

3. Could there be $F \in L^1(\mathbb{T})$ such that $\hat{F}(n) = (-1)^n$ for each $n \in \mathbb{Z}$?

4. Show that there is $F \in C(\mathbb{T})$ with $\hat{F}(n) = 1/2^n$ for $n \geq 1$ and $\hat{F}(n) = 0$ for $n \leq 0$.

5. For $F \in L^1(\mathbb{T})$, let $\sigma_N F = (1/N)(S_0 F + S_1 F + S_2 F + \cdots + S_{N-1} F)$. Note[1]. Compute $V_N F = 2\sigma_{2N} F - \sigma_N F$ in terms of the Fourier coefficients of F.
Show that $\lim_{N\to\infty} \|V_N F - F\|_1 = 0$ (for $F \in L^1(\mathbb{T})$).

6. Find the Cesàro sum of $\sum_{n=0}^{\infty} z^n$ for $z \in \mathbb{C}$, $|z| < 1$.

7. Find the Cesàro sum of $\sum_{n=0}^{\infty} z^n$ for $z \in \mathbb{C}$, $|z| = 1$, $z \neq 1$. Show also that for $z = 1$, the series $\sum_{n=0}^{\infty} z^n$ is not Cesàro summable (to a finite sum).

8. Let $G = \mathbb{R}$. What is Haar measure $\lambda_{\mathbb{R}}$? Show that the characteristic function of the unit interval $[0, 1]$ is in $L^1(\mathbb{R})$ and compute its Fourier transform.

9. For $G = \mathbb{Z}$, what is Haar measure $\lambda_{\mathbb{Z}}$? What does it mean for $F, H : \mathbb{Z} \to \mathbb{C}$ to be equal almost everywhere? How is the Fourier transform of $F \in L^1(\mathbb{Z}, \lambda_{\mathbb{Z}})$ defined?

10. For $G = \mathbb{Z}$, find an exhaustion $\mathbb{Z} = \bigcup_{n=1}^{\infty} K_n$ of $\mathbb{Z}$ by compact sets.
For the distance function $d_n : \hat{\mathbb{Z}} \times \hat{\mathbb{Z}} \to [0, \infty)$ given by
$$d_n(\chi_1, \chi_2) = \sup_{g \in K_n} |\chi_1(g) - \chi_2(g)|$$
find $\chi \in \hat{\mathbb{Z}}$ with $d_n(\chi, 1) = 0$ but $\chi \neq 1$. [Hint: $\hat{\mathbb{Z}}$ is given in Proposition 1.3.11 (iii). Here 1 stands for the trivial character, which could also be written χ_0, $\chi_0(n) = 1$ for all $n \in \mathbb{Z}$.]

Please hand in your work at the 5-6pm slot. Richard M. Timoney

[1] There should be the $S_0 F$ term to make this consistent with the notation in the notes. 8/12/207