MA342A (Harmonic Analysis 1) Tutorial sheet 7
[November 22, 2017]

Name: Solutions

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1. Show that $F \in L^2(\mathbb{T})$, $\phi \in C(\mathbb{T})$ implies $\phi * F \in L^2(\mathbb{T})$. [Hint: Combine two results we have covered.]

   Solution: We want to apply Proposition 2.4.7 in the notes with $B = L^2(\mathbb{T})$ but we need the facts that $L^2(\mathbb{T})$ is a homogeneous space of functions, that $C(\mathbb{T})$ is densely contained in $L^2(\mathbb{T})$ and that $\|F\|_2 \leq \|F\|_\infty$ for $F \in C(\mathbb{T})$. Corollary A.3.7 (in chapter 2) says that $C(\mathbb{T})$ is densely contained in $L^2(\mathbb{T})$, Examples 2.4.2 (iii) outlines that $L^2(\mathbb{T})$ is a homogeneous space.

   It remains to check that $\|F\|_2 \leq \|F\|_\infty$ for $F \in C(\mathbb{T})$ and this is because

   $$\|F\|_2^2 = \int_0^1 |F(e^{2\pi i x})|^2 \, dx \leq \int_0^1 \|F\|_\infty^2 \, dx = \|F\|_\infty^2.$$

2. Show that $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$.

   Solution: This is included in Examples 2.4.2 (iii). Starting with an $F \in L^2(\mathbb{T})$, and using the Cauchy-Schwarz inequality for $|F| \in L^2(\mathbb{T})$ together with the constant function 1 (which has $\|1\|_2 = \sqrt{\int_0^1 1^2 \, dx = 1}$), we get

   $$\langle |F|, 1 \rangle \leq \||F|\|_2 \|1\|_2 = \|F\|_2 1 \Rightarrow \|F\|_1 = \int_0^1 |F(e^{2\pi i x})| \, dx = \langle |F|, 1 \rangle \leq \|F\|_2.$$

3. Suppose $f_t \in C(\mathbb{T})$ for each $t \in [0,1]$ and that the map $t \mapsto f_t$ from $[0,1]$ to $C(\mathbb{T})$ is continuous. Show that the value at $\zeta \in \mathbb{T}$ of $\int_0^1 f_t \, dt \in C(\mathbb{T})$ is $\int_0^1 f_t(\zeta) \, dt$

   Solution: The idea here is that for $\zeta \in \mathbb{T}$, the point evaluation functional $T_\zeta \colon C(\mathbb{T}) \to \mathbb{C}$ given by the rule $T_\zeta(F) = F(\zeta)$ is a continuous linear functional. (See the proof of Proposition 2.4.6 in the notes.)

   [It is clear that $T_\zeta(F)$ is well-defined for $F \in C(\mathbb{T})$ and easy to see that it is linear — if $F, H \in C(\mathbb{T})$ and $\lambda \in \mathbb{C}$, then $T_\zeta(F + \lambda H) = (F + \lambda H)(\zeta) = F(\zeta) + \lambda H(\zeta) = T_\zeta(F) + \lambda T_\zeta(H)$. This is using the definition of the vector space operations on functions and it shows linearity. Next for $F, H \in C(\mathbb{T})$ we can see that $|T_\zeta(F) - T_\zeta(H)| = |T_\zeta(F - H)| = |(F - H)(\zeta)| \leq \|F - H\|_\infty$ and that is enough to show that T_ζ is (uniformly) continuous — given $\varepsilon > 0$, $\delta = \varepsilon$ with satisfy

   $$\|F - H\|_\infty < \delta \Rightarrow |T_\zeta(F) - T_\zeta(H)| < \varepsilon.]$$

   Then there is Lemma 2.4.5 which allows us to conclude that

   $$T_\zeta \left(\int_0^1 f_t \, dt \right) = \int_0^1 T_\zeta(f_t) \, dt = \int_0^1 f_t(\zeta) \, dt.$$

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