MA342A (Harmonic Analysis 1) Tutorial sheet 6
[November 15, 2017]

Name: Solutions

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1. By definition an action of a group G on a set X is a map $\phi\colon G \times X \to X$ such that $\phi(1_G, x) = x$ (for all $x \in X$) and $\phi(g_1 g_2, x) = \phi(g_1, \phi(g_2, x))$ (for all $g_1, g_1 \in G$, $x \in X$). [Here the group operation is written multiplicatively, $1_G \in G$ means the identity element. It can be helpful to think of $\phi(g, x)$ as $g \cdot x$ but that could well be confusing below.]

   Consider the map $\phi\colon \mathbb{T} \times C(\mathbb{T}) \to C(\mathbb{T})$ given by $\phi(\eta, F) = F_\eta$ (the translate of F by η). Show that this defines an action.

   Solution: For $G = \mathbb{T}$, we have $1_G = 1$ and, for $F \in C(\mathbb{T})$, $F_1(\zeta) = F(\zeta/1) = F(\zeta) (\forall \zeta \in \mathbb{T}) \Rightarrow F_1 = F \Rightarrow \phi(1, F) = F$.

   Next, if $\eta_1, \eta_2 \in \mathbb{T}$ and $F \in C(\mathbb{T})$, we want to verify that $\phi(\eta_1\eta_2, F) = \phi(\eta_1, \phi(\eta_2, F))$. Translating, we want $F_{\eta_1\eta_2} = (F_{\eta_2})_{\eta_1}$. For $\zeta \in \mathbb{T}$, we have

   $$F_{\eta_1\eta_2}(\zeta) = F(\zeta(\eta_1\eta_2)^{-1}) = F\left(\frac{\zeta}{\eta_1\eta_2}\right)$$

   and

   $$(F_{\eta_2})_{\eta_1}(\zeta) = (F_{\eta_2})(\zeta\eta_1^{-1}) = F(\zeta\eta_1^{-1}\eta_2^{-1}) = F\left(\frac{\zeta}{\eta_1\eta_2}\right).$$

   Hence result.

2. For $F(\zeta) = 1/(2 - \zeta^2)$, compute F_i, $(F_i)_i$, F_{-1} and F_{-i}.

   Solution:

   $$F_i(\zeta) = F(\zeta/i) = \frac{1}{2 - (\zeta/i)^2} = \frac{1}{2 - \zeta^2/i^2} = \frac{1}{2 + \zeta^2}$$

   $$(F_i)_i(\zeta) = F_i(\zeta/i) = \frac{1}{2 + (\zeta/i)^2} = \frac{1}{2 + \zeta^2/i^2} = \frac{1}{2 - \zeta^2} = F(\zeta)$$

   $$F_{-1} = F_{i^2} = (F_i)_i = F$$

   $$F_{-i} = F_{(-1)i} = (F_{-1})_i = F_i \Rightarrow F_{-i}(\zeta) = \frac{1}{2 + \zeta^2}$$

3. Show that $C(\mathbb{T}) \subseteq L^1(\mathbb{T})$ (that is $F \in C(\mathbb{T}) \Rightarrow F \in L^1(\mathbb{T})$, treating F as an equivalence class) and that $\|F\|_1 \leq \|F\|_\infty$ (for $F \in C(\mathbb{T})$).

   Solution: If we use the definition of $\mathcal{L}^1(\mathbb{T})$ as those $F \colon \mathbb{T} \to \mathbb{C}$ such that $f(x) = F(e^{2\pi i x})$ gives $f \in \mathcal{L}^1[0,1]$ (see Remark 1.3.17 in the notes), then we can say

$$F \text{ continuous} \Rightarrow f \text{ continuous (on } [0,1]) \Rightarrow f \text{ integrable.}$$

   (The reasons are that f continuous implies that its real and imaginary parts are continuous and hence both measurable. Also $|f|$ is continuous on $[0,1]$ so that the Lebesgue integral $\int_{[0,1]} |f(x)|\,dx$ agrees with the Riemann integral and is finite.)

   The passage from $\mathcal{L}^1(\mathbb{T})$ to $L^1(\mathbb{T})$ (or, equivalently from $\mathcal{L}^1[0,1]$ to $L^1[0,1]$) is just replaceing measurable functions by their almost everywhere equivalence classes.

   For $F \in C(\mathbb{T})$ we have the corresponding $f \in CP[0,1]$ and

$$\|F\|_\infty = \sup_{\zeta \in \mathbb{T}} |F(\zeta)| = \sup_{x \in [0.1]} |F(e^{2\pi i x})| = \sup_{x \in [0.1]} |f(x)| = \|f\|_\infty.$$

   Also $|f(x)| \leq \|f\|_\infty = \|F\|_\infty$ for all $x \in [0,1]$ and so

$$\|F\|_1 = \|f\|_1 = \int_{[0,1]} |f(x)|\,dx \leq \int_{[0,1]} \|F\|_\infty\,dx = \|F\|_\infty.$$

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