MA342A (Harmonic Analysis 1) Tutorial sheet 5

[November 1, 2017]

Name: Solutions

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1. Let $f \in \mathcal{L}^1[0,1]$ be $f(x) = 1$ for $0 \le x \le 1/2$ and $f(x) = 0$ for $1/2 < x \le 1$. Compute $S_0 f$, $S_1 f$ and $S_2 f$.

   Solution:

$$\hat{f}(n) = \int_{[0,1]} f(x) e^{-2\pi i n x}\, d\mu(x) = \int_{[0,1/2]} 1 e^{-2\pi i n x}\, d\mu(x) = \int_0^{1/2} e^{-2\pi i n x}\, dx$$

   (as the Lebesgue integral of a continuous function is the same as the Riemann integral). So $\hat{f}(0) = 1/2$ and for $n \neq 0$

$$\hat{f}(n) = \left[\frac{e^{-2\pi i n x}}{-2\pi i n}\right]_0^{1/2} = \frac{e^{-\pi i n}}{-2\pi i n} - \frac{1}{-2\pi i n} = \frac{(-1)^n - 1}{-2\pi i n} = \begin{cases} 0 & n \text{ even} \\ 1/(\pi i n) = -i/(\pi n) & n \text{ odd} \end{cases}$$

   So

$$\begin{aligned}
S_0 f(x) &= \hat{f}(0) e^0 = 1/2 \\
S_1 f(x) &= \hat{f}(0) + \hat{f}(-1) e^{-2\pi i x} + \hat{f}(1) e^{2\pi i x} \\
&= (1/2) + (i/\pi) e^{-2\pi i x} - (i/\pi) e^{2\pi i x} \\
&= (1/2) + 2\Re((i/\pi) e^{-2\pi i x}) \\
&= (1/2) + 2\Re((i/\pi)(\cos(-2\pi x) + i \sin(-2\pi x)) \\
&= (1/2) + 2\Re((i/\pi)(\cos 2\pi x) - i \sin(2\pi x))) \\
&= (1/2) + (2/\pi) \sin(2\pi x) \\
S_2 f(x) &= S_1 f(x) + \hat{f}(-2) e^{-4\pi i x} + \hat{f}(2) e^{4\pi i x} \\
&= S_1 f(x) \\
&= (1/2) + (2/\pi) \sin(2\pi x)
\end{aligned}$$

2. For $F \in L^1(\mathbb{T})$ and $\chi_n(\zeta) = \zeta^n$ (for $\zeta \in \mathbb{T}$, $n \in \mathbb{Z}$), compute $F * \chi_n$.

   Solution: By definition

$$(F * \chi_n)(\zeta) = \int_{\mathbb{T}} F(\eta) \chi_n(\zeta \eta^{-1})\, d\lambda(\eta)$$

and we can compute this as

$$\begin{aligned}
(F * \chi_n)(\zeta) &= \int_{\mathbb{T}} F(\eta)(\zeta\eta^{-1})^n \, d\lambda(\eta) \\
&= \int_{\mathbb{T}} F(\eta)\zeta^n\eta^{-n} \, d\lambda(\eta) \\
&= \zeta^n \int_{\mathbb{T}} F(\eta)\eta^{-n} \, d\lambda(\eta) \\
&= \zeta^n \hat{F}(n) \\
&= \hat{F}(n)\chi_n(\zeta)
\end{aligned}$$

3. For $F, H \in L^1(\mathbb{T})$, show that

$$\widehat{F * H}(n) = \hat{F}(n)\hat{H}(n)$$

[Hint: Fubini and use the fact that $\int_{\mathbb{T}} F(\zeta\eta^{-1}) \, d\lambda(\zeta) = \int_{\mathbb{T}} F(\zeta) \, d\lambda(\zeta)$ holds for $\eta \in \mathbb{T}$ (translation-invariance of the normalised arc-length measure λ on $\mathbb{T}$ and of the integral)]

Solution: By definition

$$(F * H)(\zeta) = \int_{\mathbb{T}} F(\eta)H(\zeta\eta^{-1}) \, d\lambda(\eta)$$

and

$$\widehat{F * H}(n) = \int_{\mathbb{T}} (F * H)(\zeta)\zeta^{-n} \, d\lambda(\zeta)$$

So we get an iterated integral

$$
\begin{aligned}
\widehat{F * H}(n) &= \int_{\mathbb{T}} \int_{\mathbb{T}} F(\eta) H(\zeta\eta^{-1})\, d\lambda(\eta) \zeta^{-n}\, d\lambda(\zeta) \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} F(\eta) H(\zeta\eta^{-1}) \zeta^{-n}\, d\lambda(\eta)\, d\lambda(\zeta) \\
&\quad \text{(using Fubini now)} \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} F(\eta) H(\zeta\eta^{-1}) \zeta^{-n}\, d\lambda(\zeta)\, d\lambda(\eta) \\
&= \int_{\mathbb{T}} F(\eta) \left(\int_{\mathbb{T}} H(\zeta\eta^{-1}) \zeta^{-n}\, d\lambda(\zeta) \right) d\lambda(\eta) \\
&\quad \text{(using translation invariance in the inner integral now)} \\
&= \int_{\mathbb{T}} \left(F(\eta) \int_{\mathbb{T}} H(\zeta) (\zeta\eta)^{-n}\, d\lambda(\zeta) \right) d\lambda(\eta) \\
&= \int_{\mathbb{T}} \left(F(\eta) \int_{\mathbb{T}} H(\zeta) \zeta^{-n} \eta^{-n}\, d\lambda(\zeta) \right) d\lambda(\eta) \\
&= \int_{\mathbb{T}} F(\eta) \left(\int_{\mathbb{T}} H(\zeta) \zeta^{-n}\, d\lambda(\zeta) \right) \eta^{-n}\, d\lambda(\eta) \\
&= \left(\int_{\mathbb{T}} H(\zeta) \zeta^{-n}\, d\lambda(\zeta) \right) \int_{\mathbb{T}} F(\eta) \eta^{-n}\, d\lambda(\eta) \\
&= \hat{H}(n) \hat{F}(n)
\end{aligned}
$$

Technical aside: To justify using Fubini, we need to know that $(\zeta, \eta) \mapsto F(\eta) H(\zeta\eta^{-1}) \zeta^{-n}$ is measurable on $\mathbb{T} \times \mathbb{T}$ and that it is integrable. We did not prove that $(\zeta, \eta) \mapsto H(\zeta\eta^{-1})$ is measurable, but we did use that already in showing that $F * H$ is defined. It is rather easy that $(\zeta, \eta) \mapsto F(\eta)$ and $(\zeta, \eta) \mapsto \zeta^{-n}$ are measurable, and then we have a product of measurable functions. In showing $F * H$ is defined we showed that the absolute value

$$
|F(\eta) H(\zeta\eta^{-1})|
$$

has a finite integral over $\mathbb{T} \times \mathbb{T}$ and the function we want here has the same absolute value because $|\zeta^{-n}| = 1$. So, Fubini is justifiable.

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