MA342A (Harmonic Analysis 1) Tutorial sheet 4
[October 25, 2017]

Name: Solutions

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1. Let $G = \mathbb{Z}_5$ (the cyclic group of order 5) and let $f\colon G \to \mathbb{C}$ be the function given by

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

(where we take the elements of $\mathbb{Z}_5$ to be $0, 1, 2, 3, 4$). Calculate $\hat{f}\colon \hat{G} \to \mathbb{C}$.

Solution: If we take $\zeta = e^{2\pi i/5}$ then we know that the characters of G are $\chi_j\colon G = \mathbb{Z}_5 \to \mathbb{T}$ given by

$$\chi_j(n) = \zeta^{jn}$$

($n = 0, 1, 2, 3, 4$ and $j = 0, 1, 2, 3, 4$). The formula for $\hat{f}\colon \hat{G} \to \mathbb{C}$ is

$$\hat{f}(\chi) = \frac{1}{\sqrt{5}} \sum_{g \in \mathbb{Z}_5} f(g)\overline{\chi(g)} = \frac{1}{\sqrt{5}} \sum_{n=0}^{4} f(n)\overline{\chi(n)} = \frac{1}{\sqrt{5}} f(0)\overline{\chi(0)} + 0 = \frac{1}{\sqrt{5}}.$$

So $\hat{f}$ is the constant function $1/\sqrt{5}$.

2. Consider $f\colon \mathbb{R} \to \mathbb{C}$ given by $f(x) = e^{-|x|}$. Compute $\hat{f}\colon \mathbb{R} \to \mathbb{C}$.

Solution: By definition

$$\hat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi i x t}\, dx \qquad (t \in \mathbb{R})$$

(or, maybe more correctly, that gives $\hat{f}(\chi_t)$ where $\chi_t \in \hat{\mathbb{R}}$ is given by $\chi_t(x) = e^{2\pi i x t}$).

Now we can use the dominated convergence theorem to show that

$$\hat{f}(t) = \lim_{N \to \infty} \int_{[-N,N]} f(x) e^{-2\pi i x t}\, dx$$

by taking $f_N(x) = \chi_{[-N,N]} f(x) e^{-2\pi i x t}$ (here $\chi_{[-N,N]}$ is the characteristic function of the intervale $[-N, N]$) and as dominating function $f(x) = e^{-|x|}$.

[In more detail, we certainly have $\lim_{N\to\infty} f_N(x) = f(x)e^{-2\pi i x t}$ for each $x \in \mathbb{R}$, because $f_N(x) = f(x)e^{-2\pi i x t}$ once N is fig enough that $N \geq |x|$. The functions involved are all measurable (as they are built from continuous functions and measurable characteristic functions. To show that f is integrable we can use the Monotone convergence theorem to say that

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,dx &= \lim_{N\to\infty}\int_{[-N,N]} f(x)\,dx = \lim_{N\to\infty} 2\int_0^N e^{-x}\,dx \\ &= \lim_{N\to\infty} 2[-e^{-x}]_0^N = \lim_{N\to\infty} 2(-e^{-N}+1) \\ &= 2<\infty.]\end{aligned}$$

To calculate $\int_{[-N,N]} f(x)e^{-2\pi ixt}\,dx$ we can use the Riemann integral.

$$\begin{aligned}\int_{[-N,N]} f(x)e^{-2\pi ixt}\,dx &= \int_{-N}^0 f(x)e^{-2\pi ixt}\,dx + \int_0^N f(x)e^{-2\pi ixt}\,dx \\ &= \int_{-N}^0 e^x e^{-2\pi ixt}\,dx + \int_0^N e^{-x}e^{-2\pi ixt}\,dx \\ &= \int_{-N}^0 e^{(1-2\pi it)x}\,dx + \int_0^N e^{-(1+2\pi it)x}\,dx \\ &= \left[\frac{1}{1-2\pi it}e^{(1-2\pi it)x}\right]_{-N}^0 + \left[\frac{-1}{1+2\pi it}e^{-(1+2\pi it)x}\right]_0^N \\ &= \frac{1}{1-2\pi it} - \frac{1}{1-2\pi it}e^{-(1-2\pi it)N} \\ &\quad + \frac{-1}{1+2\pi it}e^{-(1+2\pi it)N} + \frac{1}{1+2\pi it}\end{aligned}$$

Now, when we take the limit as $N\to\infty$ we see that

$$|e^{-(1\pm 2\pi it)N}| = e^{-N}|e^{\mp 2\pi it)N}| = e^{-N} = \frac{1}{e^N} \to 0.$$

Thus

$$\hat{f}(t) = \frac{1}{1-2\pi it} + \frac{1}{1+2\pi it} = \frac{(1+2\pi it)+(1-2\pi it)}{1+4\pi^2t^2} = \frac{2}{1+4\pi^2t^2}$$

Compute $\int_{\mathbb{R}} |f(x)|^2\,dx$ and $\int_{\mathbb{R}} |\hat{f}(t)|^2\,dt$. [Hint:

$$\int \frac{1}{(1+a^2t^2)^2}\,dt = \frac{t}{2(1+a^2t^2)} + \frac{1}{2a}\tan^{-1}(at) + C.]$$

Solution: By the Monotone convergence theorem

$$\begin{aligned}
\int_{\mathbb{R}} |f(x)|^2\,dx &= \lim_{N\to\infty}\int_{[-N,N]} e^{-2|x|}\,dx \\
&= \lim_{N\to\infty} 2\int_0^N e^{-2x}\,dx \\
&= \lim_{N\to\infty} 2\left[-\frac{1}{2}e^{-2x}\right]_0^N \\
&= \lim_{N\to\infty} -e^{-2N}+1 = 1.
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}} |\hat{f}(t)|^2\,dt &= \lim_{N\to\infty}\int_{[-N,N]} \frac{4}{(1+4\pi^2t^2)^2}\,dt \\
&= 4\lim_{N\to\infty}\left[\frac{t}{2(1+4\pi^2t^2)} + \frac{1}{4\pi}\tan^{-1}(2\pi t)\right]_{-N}^N \\
&= 4\lim_{N\to\infty}\Bigg(\frac{N}{2(1+4\pi^2N^2)} + \frac{1}{4\pi}\tan^{-1}(2\pi N) \\
&\qquad -\frac{-N}{2(1+4\pi^2N^2)} - \frac{1}{4\pi}\tan^{-1}(-2\pi N)\Bigg) \\
&= 4\left(0 + \frac{1}{4\pi}\frac{\pi}{2} - 0 - \frac{1}{4\pi}\left(-\frac{\pi}{2}\right)\right) \\
&= 1
\end{aligned}$$

Remark. There is a theorem we will see later called Parseval's theorem which implies that it is always true that $\int_{\mathbb{R}} |f(x)|^2\,dx = \int_{\mathbb{R}} |\hat{f}(t)|^2\,dt$, but one difficulty is to define $\hat{f}$ when we know only $\int_{\mathbb{R}} |f(x)|^2\,dx < \infty$ (and f measurable of course). So far we have needed integrability of f, that is $\int_{\mathbb{R}} |f(x)|\,dx < \infty$ and it is neither true that integrability is a consequence of square integrability ($\int_{\mathbb{R}} |f(x)|^2\,dx < \infty$) nor does integrability imply square integrability (on $\mathbb{R}$).

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