MA342A (Harmonic Analysis 1) Tutorial sheet 3

[October 18, 2017]

Name: Solutions

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1. For $G = \mathbb{Z}_4$ the cyclic group of order 4, make a table of all the values of (all) the characters $\chi \in \hat{G}$ with one row for each character and the group elements $0, 1, 2, 3$ labelling the columns.

   Solution: We write $G = \{0, 1, 2, 3\}$ with addition modulo 4.

   We know that the characters are given by $\chi(1)$ which must be a 4^{th} root of unity. That is $\chi_k(1) = e^{2\pi i k/4}$ for $k = 0, 1, 2, 3$.

   $e^{2\pi i/4} = i$ and so we get the values for $\chi_k(n) = (\chi_k(1))^n$ $(0 \leq k, n < 4)$

n	0	1	2	3
$\chi_0(n)$	1	1	1	1
$\chi_1(n)$	1	i	-1	$-i$
$\chi_2(n)$	1	-1	1	-1
$\chi_3(n)$	1	$-i$	-1	i

   Then compute the sum $\sum_{\chi \in \hat{G}} \chi$ (as a $\mathbb{C}$-valued function on G).

   Solution: $\sum_{\chi \in \hat{G}} \chi(n) = 4$ if $n = 0$ and $\sum_{\chi \in \hat{G}} \chi(n) = 0$ for $n = 1, 2, 3$.

2. Let G_1 and G_2 be finite abelian groups written additively (and with their discrete topologies). Show that the cartesian product $G_1 \times G_2$ becomes an abelian group if we define the group operation via

   $$(g_1, g_2) + (h_1, h_2) = (g_1 + h_1, g_2 + h_2).$$

   [*We write $G_1 \oplus G_2$ for this group (with the product topology, again discrete).*]

   Solution: We need to check the group properties:

   (G1) It is clear that $+$ is properly defined, that is it is a map from $(G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$.

   (G2) (associative law) For $g_1, h_1, k_1 \in G_1$ and $g_2, h_2, k_2 \in G_2$, we have to verify

   $$((g_1, g_2) + (h_1, h_2)) + (k_1, k_2) = (g_1, g_2) + ((h_1, h_2) + (k_1, k_2))$$

   But

   $$\begin{aligned} ((g_1, g_2) + (h_1, h_2)) + (k_1, k_2) &= ((g_1 + h_1) + k_1, (g_2 + h_2) + k_2) \\ &= (g_1 + (h_1 + k_1), g_2 + (h_2 + k_2)) \\ &= (g_1, g_2) + ((h_1, h_2) + (k_1, k_2)) \end{aligned}$$

   using associativity in G_1 and G_2.

(G3) (identity element) Using 0 for the identity elements of both G_1 and G_2, we see that $(0,0) \in G_1 \times G_2$ satisfies

$$(g_1, g_2) + (0,0) = (g_1 + 0, g_2 + 0) = (g_1, g_2)$$

and $(0,0) + (g_1, g_2) = (g_1, g_2)$ for each $(g_1, g_2) \in G_1 \times G_2$.

(G4) (existence of inverses) For $(g_1, g_2) \in G_1 \times G_2$, we can see that

$$(g_1, g_2) + (-g_1, -g_2) = (g_1 - g_1, g_2 - g_2) = (0,0)$$

and $(-g_1, -g_2) + (g_1, g_2) = (0,0)$. So $(-g_1, -g_2)$ is the inverse for (g_1, g_2) (where, of course, $-g_1$ is the additive inverse for $g_1 \in G_1$ and similarly for $-g_2 \in G_2$).

Finally we have an abelian group $G_1 \times G_2$ because

$$(g_1, g_2) + (h_1, h_2) = (g_1 + h_1, g_2 + h_2) = (h_1 + g_1, h_2 + g_2) = (h_1, h_2) + (g_1, g_2)$$

(since $+$ is commutative on G_1 and on G_2).

For $j = 1, 2$, define $\pi_j \colon G_1 \oplus G_2 \to G_j$ by $\pi_j(g_1, g_2) = g_j$. Show that π_j is a group homomorphism.

Solution:

$$\pi_j((g_1, g_2) + (h_1, h_2)) = \pi_j((g_1 + h_1, g_2 + h_2)) = g_j + h_j = \pi_j(g_1, g_2) + \pi_j(h_1, h_2)$$

For $j = 1, 2$, suppose $\chi_j \in \hat{G}_j$. Show that $\chi_j \circ \pi_j \in \widehat{G_1 \oplus G_2}$.

Solution: We are assuming that $\chi_j \colon G_j \to \mathbb{T}$ is a continuous homomorphism (for $j = 1, 2$) but in fact continuity imposes no restriction here because G_j has the discrete topology.

$\chi_j \circ \pi_j$ is then a composition of group homorphisms, hence a homomorphism (from $G_1 \oplus G_2$ to $\mathbb{T}$). As again $G_1 \oplus G_2$ has the discrete topology (it is a finite set and all metrics give it the discrete topology), $\chi_j \circ \pi_j$ is continuous. So $\chi_j \circ \pi_j \in \widehat{G_1 \oplus G_2}$.

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