MA342A (Harmonic Analysis 1) Tutorial sheet 2
[October 11, 2017]

Name: Solutions

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1. Let G be a finite set and d a metric on G. Show that every subset of G is open (in the metric topology).

   Solution: Fix $g \in G$ and consider all $h \in G \setminus \{g\}$. Then $d(g,h) > 0$ for all such h and there are only finitely many such h. Let $\delta_g = \min_{h \in G, h \neq g} d(g,h)$. Then ($\delta_g > 0$ and) the open ball $B(g, \delta_g)$ contains exactly one point g. So $B(g, \delta_g) = \{g\}$ is open in the metric arising from d.

   As each subset is a union of the singleton subsets it contains, it follows that each $E \subset G$ is open in the d-topology.

2. Show that every character $\chi\colon \mathbb{Z} \to \mathbb{T}$ (of the group $(\mathbb{Z}, +)$ where $\mathbb{Z}$ has the usual absolute value distance) is of the form
$$\chi_\zeta(n) = \zeta^n$$
for some $\zeta \in \mathbb{T}$.

   Solution:

   See notes, Proposition 1.3.11 (c).

3. Show that there is a group isomorphism $\mathbb{T} \to \hat{\mathbb{Z}}$ given by $\zeta \mapsto \chi_\zeta$ (with the notation above).

   Solution:

   See notes, Proposition 1.3.11 (c).

4. Let G be the group $\mathbb{Z}_2$ of the integers modulo 2 (with addition modulo 2 as the operation). Equivalently $G = \{0, 1\}$ with addition mod 2. Find $\hat{G}$.

   Solution: Say $\chi\colon \mathbb{Z}_2 \to \mathbb{T}$ is a character. That means χ is a group homomorphism from $(\mathbb{Z}_2, +)$ to $(\mathbb{T}, \cdot)$ and χ is also continuous.

   But we need a metric on $\mathbb{Z}_2$ to talk about continuity (and there is none specified in the question!). However we know from Q1 that all metrics on a finite set give the same topology, the discrete topology, where all subsets are open. (In fact defining a metric on $\{0, 1\}$ just means selecting a strictly positive distance between 0 and 1.) Since then we must give G the discrete topology, it follows that every function $\chi\colon \mathbb{Z}_2 \to \mathbb{T}$ is continuous. So continuity is not a worry here and we can forget it.

   Now if $\chi(1) = \zeta$, then $\chi(1+1) = \chi(1)\chi(1) = \zeta^2$. But $1 + 1 \equiv 0 \pmod 2$ and so we have $\chi(0) = \zeta^2$. From group theory $\chi(0) = 1$ (or we can say $\chi(0) = \chi(0+0) = \chi(0)\chi(0)$ to deduce $\chi(0) = 1$). So $\zeta = \pm 1$.

   So there are just two cases, χ_1 which has values $\chi_1(0) = \chi_1(1) = 1$ (and is the identity element of $\hat{G}$) and χ_{-1} which has values $\chi_{-1}(0) = 1$, $\chi_{-1}(1) = -1$.

So $\hat{G} = \{\chi_\zeta : \zeta = \pm 1\}$ where $\chi_\zeta(n) = \zeta_n$ for $n = 0, 1$.

As for $\mathbb{Z}$ we can multiply characters via $\chi_\zeta \chi_\eta = \chi_{\zeta\eta}$.

We can thus identify $\hat{G}$ with $\{\pm 1\}$ (a subgroup of $\mathbb{T}$ and also a cyclic group of order 2).

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